Advanced Algebra II: Teacher's Guide

By:
Kenny M. Felder

Advanced Algebra II: Teacher's Guide

By:
Kenny M. Felder

Online:
< http://cnx.org/content/col10687/1.3/ >

C O N N E X I O N S

Rice University, Houston, Texas

Collection structure revised: August 13, 2009
PDF generated: October 26, 2012
For copyright and attribution information for the modules contained in this collection, see p. 122.

Table of Contents

The Long Rambling Philosophical Introduction[1]

What you're holding in your hand is much closer to a set of detailed lesson plans than to a traditional textbook. As you read through it, your first reaction may be "Who does he think he is, telling me exactly what to say and when to say it?"

Please don't take it that way. Take it this way instead.

Over a period of time, I have developed a set of in-class assignments, homeworks, and lesson plans, that work for me and for other people who have tried them. If I give you the in-class assignments and the homeworks, but not the lesson plans, you only have 2/3 of the story; and it may not make sense without the other third. So instead, I am giving you everything: the in-class assignments and the homeworks (gathered together in the student book), the detailed explanations of all the concepts (the other student book), and the lesson plans (this document). Once you read them over, you will know exactly what I have done.

What do you do then? You may choose to follow my plan exactly, for a number of reasons—because it worked for me, or because it looks like a good plan to you, or just because you have enough other things to do without planning a lesson that I've already planned. On the other hand, you may choose to do something quite different, that incorporates my ideas in some form that I never imagined. This book is not a **proscription**, in other words, but a **resource**.

OK, with that out of the way…suppose you decide that you do want to follow my plan, exactly or pretty closely. Here's what you do.

- Right now, you read this whole introduction—despite the title, it really does contain useful information about these materials.
- Before beginning each new unit, you read my "conceptual explanation" of that unit, so you know what I'm trying to achieve.
- Each day before class, you carefully read over my lesson plan (in this document), and the in-class assignment and homework (in the student book), so you know what I'm doing and why I'm doing it.

A Typical Day in Mr. Felder's Class (…and why you care)

At the risk of repeating myself, let me emphasize—I'm not trying to insult you by suggesting that my way is the only right way to run a class. But it will help you understand these materials if you understand how I use them.

I begin each day by taking questions on **last night's** homework. I answer any and all questions. This may take five minutes, or it may take the entire class period: I don't stop until everyone is perfectly comfortable with last night's homework.

Why is that so important? Because, very often, the homework introduces new concepts that the students have **never seen in class before**. For instance, very early in the first unit, I introduce the idea of "permuting" graphs: for instance, if you add 3 to any function, the graph moves up by three units. This

[1] This content is available online at <http://cnx.org/content/m19503/1.1/>.

concept never comes up in class, in any form—it is developed entirely on a homework. So it's vitally important to debrief them the next day and make sure that they got, not only the right answers, but the point.

After the homework is covered, I begin a new topic. This is almost (almost!) never done in a long lecture. Sometimes it happens in a class discussion; sometimes it happens in a TAPPS exercise (more on that when we do our first one); most often, it happens in an **in-class assignment**. These assignments should almost always be done in pairs or groups of three, very rarely individually. They generally require pretty high-level thinking. On a good day I can hear three or four heated arguments going on in different groups. Most of my class time is spent moving between different groups and helping them when they are stuck. In general, there is some particular **point** I want them to get from the exercise, and they will need that point to do the homework—so a lot of my job in class is to make sure that, before they leave, they got the point.

Timing

If you read through this entire document (which I do not recommend at one sitting), you get the illusion that I have everything planned down to the day. If I say "do this assignment in class, then do this homework," they had better get that done in one day, or they will fall irretrievably behind.

Well, suppose you add it all up that way. Every "1-day assignment" (with homework) counts as one day, and what the heck, let's allocate two days for every test (one day for preparation, using the "Sample Test"—and one day for the actual test). If you add it up that way, you will get a total of 91 days, or thereabouts. There are 180 days in the school year.

So what does that mean? Does it mean you will be done in one semester? No, of course not. It means, take your time and do it right.

For one thing, I believe in building in a lot of time for review. Ideally, two weeks before mid-terms and another two weeks before finals. (What I do during this time is cover one topic a day, with the students teaching each class.)

But even leaving that aside, one apparent day's worth of material will sometimes take you two days to get through. You spend the whole day reviewing last night's homework and you don't even get to the new assignment. Or, you get to the end of the class and you realize that most of the groups are only half-way through the in-class assignment. Don't rush it! It's much more important to get today's concept, and really make sure everyone has it, then to rush on to tomorrow. The way I see it, you have three reasonable choices.

1. If **most** of the class is **mostly** finished with the in-class assignment, it may make sense to say "Finish the in-class assignment tonight, and also do the homework."
2. If most of the class is only half-way done, it may make sense to say "Finish the in-class assignment tonight, and we will do the homework in class tomorrow." This puts you a half-day "behind" which is fine. **However, some in-class assignments really cannot be done at home**...they require too much group work or help from you. So...
3. Sometimes you just say "We'll finish the in-class assignment tomorrow." This puts you one day "behind" which is also fine.

Of course you need to pace yourself. But do it by tests, not by days. There are sixteen tests. If you are going at a clip that will get you through more or less that many tests by the end of the year, you're doing fine. And even that isn't exact—of course, some units will take longer than others. Personally, I would much rather skip the unit on Conics (the last unit) entirely, than lose the entire class by trying to rush through Exponents. (However, in real life, I do make it through the entire syllabus.)

Tests

At the end of every unit I have a "Sample Test." This is for the students' benefit as much as for yours: it makes a great study guide and/or homework. If you say "The homework tonight is the sample test.

Tomorrow we will go over any questions you have on the sample test, and on the topic in general—that will be your last chance to ask me questions! The next day will be our actual test," then you are giving the students a great chance to bone up before the test. Doing this has dramatically improved my classes.

So what about your actual test? Of course, you may (or may not) want to **base** your test on mine. In that case, however, be careful about timing—some of my "Sample Tests" are actually too long to be a real test. But they are made up of actual questions that I have used on actual tests in the past—and in any case, calling them "Sample Tests" gets students' attention better than calling them "Review Questions."

By the way, although I do not generally recommend using **exactly** my questions—you want to change the numbers at least—it is sometimes OK to use **exactly** my extra credit. Even if they just did it, it often has enough real learning in it that it is worth giving them a few points if they took the time to look it over and/or ask about it.

How to Use Advanced Algebra II[2]

Over a period of time, I have developed a set of in-class assignments, homeworks, and lesson plans, that work for me and for other people who have tried them. The complete set comprises three separate books that work together:

- The Homework and Activities Book[3] contains in-class and homework assignments that are given to the students day-by-day.
- The Concepts Book[4] provides conceptual explanations, and is intended as a reference or review guide for students; it is not used when teaching the class.
- The Teacher's Guide[5] provides lesson plans; it is your guide to how I envisioned these materials being used when I created them (and how I use them myself).

Instructors should note that this book probably contains more information than you will be able to cover in a single school year. I myself do not teach from every chapter in my own classes, but have chosen to include these additional materials to assist you in meeting your own needs. As you will likely need to cut some sections from the book, I strongly recommend that you spend time early on to determine which modules are most important for your state requirements and personal teaching style.

One more warning is important: these materials were designed for an **Advanced** Algebra II course. For such a course, I hope this will provide you with ready-to-use textbook and lesson plans. If you are teaching a Standard or Remedial-level course, these materials will still be useful, but you will probably have to cut or reduce some of the most conceptual material, and supplement it with more drill-and-practice than I provide.

The following table of contents provides a list of topics covered in this course with links to each module. You can use these links to move between the books or to jump ahead to any topic.

	Conceptual Explanations [col10624][644]	**Activities & Homework [col10686]**[645]	**Teacher's Guide [col10687]**
Functions			
Introduction			[m19335][646]
Function Concepts	[m18192][647]		
What is a Variable?	[m18194][648]		
What is a Function?	[m18189][649]		

continued on next page

[2]This content is available online at <http://cnx.org/content/m19435/1.6/>.
[3]*Advanced Algebra II: Activities and Homework* <http://cnx.org/content/col10686/latest/>
[4]*Advanced Algebra II: Conceptual Explanations* <http://cnx.org/content/col10624/latest/>
[5]*Advanced Algebra II: Teacher's Guide* <http://cnx.org/content/col10687/latest/>

The Rule of Consistency	[m19190][650]		
Four Ways to Represent a Function	[m18195][651]		
Domain and Range	[m18191][652]		
The Function Game		[m19121][653]	[m19342][654]
The Function Game: Introduction		[m19125][655]	
The Function Game: Leader's Sheet		[m19126][656]	
The Function Game: Answer Sheet		[m19124][657]	
Functions in the Real World	[m18193][658]		[m19331][659]
Homework: Functions in the Real World		[m19115][660]	
Function Notation	[m18188][661]		
Algebraic Generalizations	[m18186][662]	[m19114][663]	[m19332][664]
Homework: Algebraic Generalizations		[m19108][665]	
Graphing	[m18196][666]		[m19334][667]
Homework: Graphing		[m19116][668]	
Horizontal and Vertical Permutations		[m19110][669]	[m19339][670]
Homework: Horizontal and Vertical Permutations I		[m19119][671]	
Homework: Horizontal and Vertical Permutations II		[m31952][672]	
Sample Test: Functions I		[m19122][673]	[m19340][674]
Lines	[m18197][675]	[m19113][676]	[m19337][677]

continued on next page

Homework: Graphing Lines		[m19118][678]	
Composite Functions	[m18187][679]	[m19109][680]	[m19333][681]
Homework: Composite Functions		[m19107][682]	
Inverse Functions	[m18198][683]	[m19112][684]	[m19336][685]
Homework: Inverse Functions		[m19120][686]	
TAPPS Exercise: How do I Solve That For y?		[m19123][687]	
Sample Test: Functions II		[m19117][688]	
Inequalities and Absolute Values			
Introduction			[m19432][689]
Inequalities	[m18205][690]	[m19158][691]	[m19430][692]
Homework: Inequalities		[m19154][693]	
Inequality Word Problems		[m19163][694]	[m19428][695]
Absolute Value Equations	[m18201][696]	[m19148][697]	[m19426][698]
Homework: Absolute Value Equations		[m19151][699]	
Absolute Value Inequalities	[m18207][700]	[m19151][701]	[m19431][702]
Homework: Absolute Value Inequalities		[m19155][703]	
Graphing Absolute Values	[m18199][704]		
Graphing Inequalities	[m18208][705]		
Graphing Inequalities and Absolute Values		[m19150][706]	[m19433][707]

continued on next page

"Piecewise Functions" and Absolute Value	[m18200][708]		
Homework: Graphing Inequalities and Absolute Values		[m19153][709]	
Sample Test: Inequalities and Absolute Values		[m19166][710]	
Simultaneous Equations			
Introduction to Simultaneous Equations			[m19497][711]
Distance, Rate, and Time	[m18211][712]	[m19288][713]	
Simultaneous Equations by Graphing	[m18209][714]	[m19291][715]	
Homework: Simultaneous Equations by Graphing		[m19291][716]	
Substitution	[m18211][717]		
Elimination	[m18215][718]		
Special Cases	[m18213][719]		
Word Problems	[m18210][720]		
Using Letters as Numbers	[m18214][721]		
Simultaneous Equations		[m19293][722]	[m19498][723]
Homework: Simultaneous Equations		[m19289][724]	
The "Generic" Simultaneous Equation		[m19294][725]	[m19499][726]
Sample Test: 2 Equations and 2 Unknowns		[m19292][727]	
Quadratics			
Introduction			[m19469][728]
continued on next page			

Multiplying Binomials	[m18224][729]	[m19247][730]	[m19472][731]
Homework: Multiplying Binomials		[m19253][732]	
Factoring	[m18227][733]	[m19243][734]	[m19466][735]
Solving Quadratic Equations by Factoring	[m18222][736]		
Homework: Factoring Expressions		[m19248][737]	
Introduction to Quadratic Equations		[m19246][738]	[m19470][739]
Homework: Introduction to Quadratic Equations		[m19251][740]	
Solving Quadratic Equations by Completing the Square	[m18217][741]		
Completing the Square		[m19242][742]	[m19465][743]
Homework: Completing the Square		[m19249][744]	
The Quadratic Formula	[m18231][745]		
The "Generic" Quadratic Equation		[m19262][746]	[m19480][747]
Homework: Solving Quadratic Equations		[m19256][748]	
Sample Test: Quadratic Equations I		[m19259][749]	
Different Types of Solutions to Quadratic Equations	[m18216][750]		
Graphing Quadratic Functions	[m18228][751]	[m19245][752]	[m19468][753]
Graphing Quadratic Functions II		[m19244][754]	[m19467][755]

continued on next page

Homework: Graphing Quadratic Functions II		[m19250][756]	
Solving Problems by Graphing Quadratic Equations	[m18220][757]	[m19260][758]	[m19479][759]
Homework: Solving Problems by Graphing Quadratic Equations		[m19255][760]	
Quadratic Inequalities	[m18230][761]	[m19257][762]	
Homework: Quadratic Inequalities		[m19254][763]	[m19473][764]
Sample Test: Quadratics II		[m19258][765]	
Exponents			
Introduction			[m19325][766]
Exponent Concepts	[m18232][767]		
Laws of Exponents	[m18235][768]		
Zero, Negative Numbers, and Fractions as Exponents	[m18234][769]		
Exponential Curves	[m18233][770]		
Rules of Exponents		[m19104][771]	[m19327][772]
Homework: Rules of Exponents		[m19101][773]	
Extending the Idea of Exponents		[m19096][774]	[m19328][775]
Homework: Extending the Idea of Exponents		[m19098][776]	
Fractional Exponents		[m19097][777]	[m19322][778]
Homework: Fractional Exponents		[m19100][779]	
"Real Life" Exponential Curves		[m19103][780]	[m19329][781]
			continued on next page

Homework: "Real Life" Exponential Curves		[m19102][782]	
Sample Test: Exponents		[m19105][783]	
Logarithms			
Logarithm Concepts	[m18242][784]		
Logarithms Explained by Analogy to Roots	[m18236][785]		
Rewriting Logarithm Equations as Exponent Equations	[m18241][786]		
The Logarithm Defined as an Inverse Function	[m18240][787]		
Introduction		[m19175][788]	[m19436][789]
Homework: Logs		[m19176][790]	
Properties of Logarithms	[m18239][791]	[m19269][792]	[m19438][793]
Homework: Properties of Logarithms		[m19177][794]	
Using the Laws of Logarithms		[m19184][795]	[m19440][796]
Common Logarithms	[m18237][797]		
Graphing Logarithmic Functions	[m18238][798]		
So What Are Logarithms Good For, Anyway?		[m19181][799]	
Homework: So What Are Logarithms Good For, Anyway?		[m19268][800]	[m19439][801]
Sample Test		[m19180][802]	
Rational Expressions			
Introduction			[m19486][803]

continued on next page

Rational Expressions Concepts	[m18304][804]		
Simplifying Rational Expressions	[m18296][805]		
Multiplying Rational Expressions	[m18301][806]		
Adding and Subtracting Rational Expressions	[m18303][807]		
Rational Expressions		[m19278][808]	[m19488][809]
Homework: Rational Expressions		[m19275][810]	
Rational Equations	[m18302][811]	[m19279][812]	[m19489][813]
Homework: Rational Expressions and Equations		[m19277][814]	
Dividing Polynomials	[m18299][815]	[m19276][816]	[m19487][817]
Sample Test: Rational Expressions		[m19274][818]	
Radicals			
Radical Concepts	[m18244][819]		
Radicals (*aka Roots)		[m19420][820]	
Properties of Radicals	[m18271][821]		
Radicals and Exponents		[m19419][822]	
Some Very Important Generalizations		[m19422][823]	
Simplifying Radicals	[m18274][824]	[m19421][825]	
Introduction			[m19484][826]
Homework: Radicals		[m19270][827]	
A Bunch of Other Stuff About Radicals		[m19263][828]	[m19483][829]
Homework: A Bunch of Other Stuff About Radicals		[m19264][830]	

continued on next page

Radical Equations	[m18273][831]	[m19272][832]	[m19485][833]
Homework: Radical Equations		[m19271][834]	
Sample Test: Radicals		[m19273][835]	
Imaginary Numbers			
Introduction			[m19424][836]
Imaginary Numbers Concepts	[m18285][837]		
Playing with i	[m18286][838]		
Introduction to Imaginary Numbers			[m21990][839]
Imaginary Numbers		[m19129][840]	
Homework: Imaginary Numbers		[m19130][841]	
Complex Numbers	[m18282][842]	[m19128][843]	[m19423][844]
Equality and Inequality in Complex Numbers	[m18283][845]		
Homework: Complex Numbers		[m19132][846]	
Quadratic Equations and Complex Numbers	[m18288][847]		
Me, Myself, and the Square Root of i		[m19134][848]	[m19425][849]
The Many Merry Cube Roots of -1		[m19131][850]	
Homework: Quadratic Equations and Complex Numbers		[m19127][851]	
A Few "Extras For Experts" Thoughts on Imaginary Numbers	[m18284][852]		
Sample Test: Complex Numbers		[m19133][853]	
continued on next page			

Matrices			
Matrices	[m18311][854]		
Introduction to Matrices		[m19206][855]	[m19445][856]
Homework: Introduction to Matrices		[m19205][857]	
Multiplying Matrices	[m18291][858]		
Multiplying Matrices I		[m19207][859]	[m19448][860]
Homework: Multiplying Matrices I		[m19196][861]	
Multiplying Matrices II		[m19208][862]	[m19449][863]
Homework: Multiplying Matrices II		[m19201][864]	
The Identity Matrix	[m18293][865]		
The Inverse Matrix	[m18294][866]		
The Identity and Inverse Matrices		[m19213][867]	[m19443][868]
Homework: The Identity and Inverse Matrices		[m19194][869]	
The Inverse of the Generic 2x2 Matrix		[m19214][870]	[m19446][871]
Using Matrices for Transformations		[m19221][872]	[m19451][873]
Homework: Using Matrices for Transformations		[m19190][874]	
Sample Test: Matrices I		[m19210][875]	
Matrices on a TI-83 or TI-84 Calculator	[m18290][876]		
Matrices on the Calculator			[m19447][877]

continued on next page

Homework: Calculators		[m19188][878]	
Determinants	[m18289][879]		[m19442][880]
Homework: Determiners		[m19193][881]	
Solving Linear Equations	[m18292][882]	[m19212][883]	[m19450][884]
Homework: Solving Linear Equations		[m19204][885]	
Sample Test: Matrices II		[m19209][886]	
Modeling Data with Functions			
Introduction			[m19454][887]
Data Modeling Concepts	[m18277][888]		
Direct and Inverse Variation	[m18281][889]		[m19452][890]
Direct Variation		[m19228][891]	
Homework: Inverse Variation		[m19227][892]	
Homework: Direct and Inverse Variation		[m19225][893]	
Finding a Linear Function For Any Two Points	[m18278][894]		
Finding a Parabolic Function For Any Three Points	[m18279][895]		
From Data Points to Functions		[m19224][896]	
Homework: From Data Points to Functions		[m19232][897]	
Calculator Regression	[m18280][898]		[m19453][899]
Homework: Calculator Regression		[m19231][900]	
			continued on next page

Sample Test: Modeling Data With Functions		[m19222][901]	
Conics			
Introduction			[m19307][902]
Conic Concepts	[m18265][903]		
A Mathematical Look at Distance	[m18246][904]		
Distance		[m19081][905]	[m19299][906]
Homework: Distance		[m19086][907]	
Circles	[m18245][908]		[m19298][909]
All the Points Equidistant from a Given Point		[m19078][910]	
Homework: Circles		[m19084][911]	
Parabolas	[m18268][912]		
All the Points Equidistant from a Point and a Line		[m19079][913]	
Parabolas: Day 1			[m19315][914]
Homework: Vertical and Horizontal Parabolas		[m19086][915]	
Parabolas: Day 2			[m19313][916]
Parabolas: From Definition to Equation		[m19092][917]	[m19311][918]
Sample Test: Distance, Circles, and Parabolas		[m19094][919]	
Ellipses	[m18247][920]		[m19303][921]
Distance from this point plus distance to that point is Constant		[m19083][922]	
Homework: Ellipses		[m19088][923]	
Ellipses: From Definition to Equation		[m19095][924]	[m19305][925]

continued on next page

Hyperbolas	[m18249][926]		[m19306][927]
Distance from this point minus distance from that point is constant		[m19082][928]	
Homework: Hyperbolas		[m19089][929]	
A Brief Recap: How Do You Tell What Shape It Is?	[m18270][930]		
Sample Test: Conics 2 (Ellipses and Hyperbolas)		[m19093][931]	
Sequences and Series			
Prerequisites			[m19495][932]
Sequences	[m19076][933]		
Arithmetic and Geometric Sequences		[m19285][934]	[m19490][935]
Homework: Arithmetic and Geometric Sequences		[m19284][936]	
Series	[m19074][937]		
Series and Series Notation			[m19491][938]
Homework: Series and Series Notation		[m19280][939]	
Arithmetic and Geometric Series			[m19494][940]
Homework: Arithmetic and Geometric Series		[m19282][941]	
Proof by Induction	[m19075][942]		[m19492][943]
Homework: Proof by Induction		[m19281][944]	
Extra Credit			[m19493][945]
Sample Test: Sequences and Series		[m19283][946]	

continued on next page

Probability			
How Many Groups?		[m19236][947]	
Tree Diagrams			[m19463][948]
Homework: Tree Diagrams		[m19234][949]	
Probability Concepts	[m19073][950]		
Introduction to Probability		[m19237][951]	[m19461][952]
Homework: The Multiplication Rule		[m19233][953]	
Trickier Probability Problems			[m19464][954]
Homework: Trickier Probability Problems		[m19235][955]	
Permutations	[m19072][956]		[m19462][957]
Homework: Permutations		[m19241][958]	
Combinations	[m19071][959]		[m19460][960]
Homework: Permutations and Combinations		[m19240][961]	
Sample Test: Probability		[m19238][962]	

Table 1

[644]*Advanced Algebra II: Conceptual Explanations* <http://cnx.org/content/col10624/latest/>
[645]*Advanced Algebra II: Activities and Homework* <http://cnx.org/content/col10686/latest/>
[646]http://cnx.org/content/m19335/latest/?collection=col10687
[647]http://cnx.org/content/m18192/latest/?collection=col10624
[648]http://cnx.org/content/m18194/latest/?collection=col10624
[649]http://cnx.org/content/m18189/latest/?collection=col10624
[650]http://cnx.org/content/m18190/latest/?collection=col10624
[651]http://cnx.org/content/m18195/latest/?collection=col10624
[652]http://cnx.org/content/m18191/latest/?collection=col10624
[653]http://cnx.org/content/m19121/latest/?collection=col10686
[654]http://cnx.org/content/m19342/latest/?collection=col10687
[655]http://cnx.org/content/m19125/latest/?collection=col10686
[656]http://cnx.org/content/m19126/latest/?collection=col10686
[657]http://cnx.org/content/m19124/latest/?collection=col10686
[658]http://cnx.org/content/m18193/latest/?collection=col10624
[659]http://cnx.org/content/m19331/latest/?collection=col10687
[660]http://cnx.org/content/m19115/latest/?collection=col10686
[661]http://cnx.org/content/m18188/latest/?collection=col10624
[662]http://cnx.org/content/m18186/latest/?collection=col10624
[663]http://cnx.org/content/m19114/latest/?collection=col10686
[664]http://cnx.org/content/m19332/latest/?collection=col10687
[665]http://cnx.org/content/m19108/latest/?collection=col10686
[666]http://cnx.org/content/m18196/latest/?collection=col10624
[667]http://cnx.org/content/m19334/latest/?collection=col10687
[668]http://cnx.org/content/m19116/latest/?collection=col10686
[669]http://cnx.org/content/m19110/latest/?collection=col10686
[670]http://cnx.org/content/m19339/latest/?collection=col10687
[671]http://cnx.org/content/m19119/latest/?collection=col10686
[672]http://cnx.org/content/m31952/latest/?collection=col10686
[673]http://cnx.org/content/m19122/latest/?collection=col10686
[674]http://cnx.org/content/m19340/latest/?collection=col10687
[675]http://cnx.org/content/m18197/latest/?collection=col10624
[676]http://cnx.org/content/m19113/latest/?collection=col10686
[677]http://cnx.org/content/m19337/latest/?collection=col10687
[678]http://cnx.org/content/m19118/latest/?collection=col10686
[679]http://cnx.org/content/m18187/latest/?collection=col10624
[680]http://cnx.org/content/m19109/latest/?collection=col10686
[681]http://cnx.org/content/m19333/latest/?collection=col10687
[682]http://cnx.org/content/m19107/latest/?collection=col10686
[683]http://cnx.org/content/m18198/latest/?collection=col10624
[684]http://cnx.org/content/m19112/latest/?collection=col10686
[685]http://cnx.org/content/m19336/latest/?collection=col10687
[686]http://cnx.org/content/m19120/latest/?collection=col10686
[687]http://cnx.org/content/m19123/latest/?collection=col10686
[688]http://cnx.org/content/m19117/latest/?collection=col10686
[689]http://cnx.org/content/m19432/latest/?collection=col10687
[690]http://cnx.org/content/m18205/latest/?collection=col10624
[691]http://cnx.org/content/m19158/latest/?collection=col10686
[692]http://cnx.org/content/m19430/latest/?collection=col10687
[693]http://cnx.org/content/m19154/latest/?collection=col10686
[694]http://cnx.org/content/m19163/latest/?collection=col10686
[695]http://cnx.org/content/m19428/latest/?collection=col10687
[696]http://cnx.org/content/m18201/latest/?collection=col10624
[697]http://cnx.org/content/m19148/latest/?collection=col10686
[698]http://cnx.org/content/m19426/latest/?collection=col10687
[699]http://cnx.org/content/m19151/latest/?collection=col10686
[700]http://cnx.org/content/m18207/latest/?collection=col10624
[701]http://cnx.org/content/m19149/latest/?collection=col10686
[702]http://cnx.org/content/m19431/latest/?collection=col10687
[703]http://cnx.org/content/m19155/latest/?collection=col10686
[704]http://cnx.org/content/m18199/latest/?collection=col10624

[705]http://cnx.org/content/m18208/latest/?collection=col10624
[706]http://cnx.org/content/m19150/latest/?collection=col10686
[707]http://cnx.org/content/m19433/latest/?collection=col10687
[708]http://cnx.org/content/m18200/latest/?collection=col10624
[709]http://cnx.org/content/m19153/latest/?collection=col10686
[710]http://cnx.org/content/m19166/latest/?collection=col10686
[711]http://cnx.org/content/m19497/latest/?collection=col10687
[712]http://cnx.org/content/m18211/latest/?collection=col10624
[713]http://cnx.org/content/m19288/latest/?collection=col10686
[714]http://cnx.org/content/m18209/latest/?collection=col10624
[715]http://cnx.org/content/m19291/latest/?collection=col10686
[716]http://cnx.org/content/m19291/latest/?collection=col10686
[717]http://cnx.org/content/m18211/latest/?collection=col10624
[718]http://cnx.org/content/m18215/latest/?collection=col10624
[719]http://cnx.org/content/m18213/latest/?collection=col10624
[720]http://cnx.org/content/m18210/latest/?collection=col10624
[721]http://cnx.org/content/m18214/latest/?collection=col10624
[722]http://cnx.org/content/m19293/latest/?collection=col10686
[723]http://cnx.org/content/m19498/latest/?collection=col10687
[724]http://cnx.org/content/m19289/latest/?collection=col10686
[725]http://cnx.org/content/m19294/latest/?collection=col10686
[726]http://cnx.org/content/m19499/latest/?collection=col10687
[727]http://cnx.org/content/m19292/latest/?collection=col10686
[728]http://cnx.org/content/m19469/latest/?collection=col10687
[729]http://cnx.org/content/m18224/latest/?collection=col10624
[730]http://cnx.org/content/m19247/latest/?collection=col10686
[731]http://cnx.org/content/m19472/latest/?collection=col10687
[732]http://cnx.org/content/m19253/latest/?collection=col10686
[733]http://cnx.org/content/m18227/latest/?collection=col10624
[734]http://cnx.org/content/m19243/latest/?collection=col10686
[735]http://cnx.org/content/m19466/latest/?collection=col10687
[736]http://cnx.org/content/m18222/latest/?collection=col10624
[737]http://cnx.org/content/m19248/latest/?collection=col10686
[738]http://cnx.org/content/m19246/latest/?collection=col10686
[739]http://cnx.org/content/m19470/latest/?collection=col10687
[740]http://cnx.org/content/m19251/latest/?collection=col10686
[741]http://cnx.org/content/m18217/latest/?collection=col10624
[742]http://cnx.org/content/m19242/latest/?collection=col10686
[743]http://cnx.org/content/m19465/latest/?collection=col10687
[744]http://cnx.org/content/m19249/latest/?collection=col10686
[745]http://cnx.org/content/m18231/latest/?collection=col10624
[746]http://cnx.org/content/m19262/latest/?collection=col10686
[747]http://cnx.org/content/m19480/latest/?collection=col10687
[748]http://cnx.org/content/m19256/latest/?collection=col10686
[749]http://cnx.org/content/m19259/latest/?collection=col10686
[750]http://cnx.org/content/m18216/latest/?collection=col10624
[751]http://cnx.org/content/m18228/latest/?collection=col10624
[752]http://cnx.org/content/m19245/latest/?collection=col10686
[753]http://cnx.org/content/m19468/latest/?collection=col10687
[754]http://cnx.org/content/m19244/latest/?collection=col10686
[755]http://cnx.org/content/m19467/latest/?collection=col10687
[756]http://cnx.org/content/m19250/latest/?collection=col10686
[757]http://cnx.org/content/m18220/latest/?collection=col10624
[758]http://cnx.org/content/m19260/latest/?collection=col10686
[759]http://cnx.org/content/m19479/latest/?collection=col10687
[760]http://cnx.org/content/m19255/latest/?collection=col10686
[761]http://cnx.org/content/m18230/latest/?collection=col10624
[762]http://cnx.org/content/m19257/latest/?collection=col10686
[763]http://cnx.org/content/m19254/latest/?collection=col10686
[764]http://cnx.org/content/m19473/latest/?collection=col10687
[765]http://cnx.org/content/m19258/latest/?collection=col10686

[766]http://cnx.org/content/m19325/latest/?collection=col10687
[767]http://cnx.org/content/m18232/latest/?collection=col10624
[768]http://cnx.org/content/m18235/latest/?collection=col10624
[769]http://cnx.org/content/m18234/latest/?collection=col10624
[770]http://cnx.org/content/m18233/latest/?collection=col10624
[771]http://cnx.org/content/m19104/latest/?collection=col10686
[772]http://cnx.org/content/m19327/latest/?collection=col10687
[773]http://cnx.org/content/m19101/latest/?collection=col10686
[774]http://cnx.org/content/m19096/latest/?collection=col10686
[775]http://cnx.org/content/m19328/latest/?collection=col10687
[776]http://cnx.org/content/m19098/latest/?collection=col10686
[777]http://cnx.org/content/m19097/latest/?collection=col10686
[778]http://cnx.org/content/m19322/latest/?collection=col10687
[779]http://cnx.org/content/m19100/latest/?collection=col10686
[780]http://cnx.org/content/m19103/latest/?collection=col10686
[781]http://cnx.org/content/m19329/latest/?collection=col10687
[782]http://cnx.org/content/m19102/latest/?collection=col10686
[783]http://cnx.org/content/m19105/latest/?collection=col10686
[784]http://cnx.org/content/m18242/latest/?collection=col10624
[785]http://cnx.org/content/m18236/latest/?collection=col10624
[786]http://cnx.org/content/m18241/latest/?collection=col10624
[787]http://cnx.org/content/m18240/latest/?collection=col10624
[788]http://cnx.org/content/m19175/latest/?collection=col10686
[789]http://cnx.org/content/m19436/latest/?collection=col10687
[790]http://cnx.org/content/m19176/latest/?collection=col10686
[791]http://cnx.org/content/m18239/latest/?collection=col10624
[792]http://cnx.org/content/m19269/latest/?collection=col10686
[793]http://cnx.org/content/m19438/latest/?collection=col10687
[794]http://cnx.org/content/m19177/latest/?collection=col10686
[795]http://cnx.org/content/m19184/latest/?collection=col10686
[796]http://cnx.org/content/m19440/latest/?collection=col10687
[797]http://cnx.org/content/m18237/latest/?collection=col10624
[798]http://cnx.org/content/m18238/latest/?collection=col10624
[799]http://cnx.org/content/m19181/latest/?collection=col10686
[800]http://cnx.org/content/m19268/latest/?collection=col10686
[801]http://cnx.org/content/m19439/latest/?collection=col10687
[802]http://cnx.org/content/m19180/latest/?collection=col10686
[803]http://cnx.org/content/m19486/latest/?collection=col10687
[804]http://cnx.org/content/m18304/latest/?collection=col10624
[805]http://cnx.org/content/m18296/latest/?collection=col10624
[806]http://cnx.org/content/m18301/latest/?collection=col10624
[807]http://cnx.org/content/m18303/latest/?collection=col10624
[808]http://cnx.org/content/m19278/latest/?collection=col10686
[809]http://cnx.org/content/m19488/latest/?collection=col10687
[810]http://cnx.org/content/m19275/latest/?collection=col10686
[811]http://cnx.org/content/m18302/latest/?collection=col10624
[812]http://cnx.org/content/m19279/latest/?collection=col10686
[813]http://cnx.org/content/m19489/latest/?collection=col10687
[814]http://cnx.org/content/m19277/latest/?collection=col10686
[815]http://cnx.org/content/m18299/latest/?collection=col10624
[816]http://cnx.org/content/m19276/latest/?collection=col10686
[817]http://cnx.org/content/m19487/latest/?collection=col10687
[818]http://cnx.org/content/m19274/latest/?collection=col10686
[819]http://cnx.org/content/m18244/latest/?collection=col10624
[820]http://cnx.org/content/m19420/latest/?collection=col10686
[821]http://cnx.org/content/m18271/latest/?collection=col10624
[822]http://cnx.org/content/m19419/latest/?collection=col10686
[823]http://cnx.org/content/m19422/latest/?collection=col10686
[824]http://cnx.org/content/m18274/latest/?collection=col10624
[825]http://cnx.org/content/m19421/latest/?collection=col10686
[826]http://cnx.org/content/m19484/latest/?collection=col10687

[827]http://cnx.org/content/m19270/latest/?collection=col10686
[828]http://cnx.org/content/m19263/latest/?collection=col10686
[829]http://cnx.org/content/m19483/latest/?collection=col10687
[830]http://cnx.org/content/m19264/latest/?collection=col10686
[831]http://cnx.org/content/m18273/latest/?collection=col10624
[832]http://cnx.org/content/m19272/latest/?collection=col10686
[833]http://cnx.org/content/m19485/latest/?collection=col10687
[834]http://cnx.org/content/m19271/latest/?collection=col10686
[835]http://cnx.org/content/m19273/latest/?collection=col10686
[836]http://cnx.org/content/m19424/latest/?collection=col10687
[837]http://cnx.org/content/m18285/latest/?collection=col10624
[838]http://cnx.org/content/m18286/latest/?collection=col10624
[839]http://cnx.org/content/m21990/latest/?collection=col10687
[840]http://cnx.org/content/m19129/latest/?collection=col10686
[841]http://cnx.org/content/m19130/latest/?collection=col10686
[842]http://cnx.org/content/m18282/latest/?collection=col10624
[843]http://cnx.org/content/m19128/latest/?collection=col10686
[844]http://cnx.org/content/m19423/latest/?collection=col10687
[845]http://cnx.org/content/m18283/latest/?collection=col10624
[846]http://cnx.org/content/m19132/latest/?collection=col10686
[847]http://cnx.org/content/m18288/latest/?collection=col10624
[848]http://cnx.org/content/m19134/latest/?collection=col10686
[849]http://cnx.org/content/m19425/latest/?collection=col10687
[850]http://cnx.org/content/m19131/latest/?collection=col10686
[851]http://cnx.org/content/m19127/latest/?collection=col10686
[852]http://cnx.org/content/m18284/latest/?collection=col10624
[853]http://cnx.org/content/m19133/latest/?collection=col10686
[854]http://cnx.org/content/m18311/latest/?collection=col10624
[855]http://cnx.org/content/m19206/latest/?collection=col10686
[856]http://cnx.org/content/m19445/latest/?collection=col10687
[857]http://cnx.org/content/m19205/latest/?collection=col10686
[858]http://cnx.org/content/m18291/latest/?collection=col10624
[859]http://cnx.org/content/m19207/latest/?collection=col10686
[860]http://cnx.org/content/m19448/latest/?collection=col10687
[861]http://cnx.org/content/m19196/latest/?collection=col10686
[862]http://cnx.org/content/m19208/latest/?collection=col10686
[863]http://cnx.org/content/m19449/latest/?collection=col10687
[864]http://cnx.org/content/m19201/latest/?collection=col10686
[865]http://cnx.org/content/m18293/latest/?collection=col10624
[866]http://cnx.org/content/m18294/latest/?collection=col10624
[867]http://cnx.org/content/m19213/latest/?collection=col10686
[868]http://cnx.org/content/m19443/latest/?collection=col10687
[869]http://cnx.org/content/m19194/latest/?collection=col10686
[870]http://cnx.org/content/m19214/latest/?collection=col10686
[871]http://cnx.org/content/m19446/latest/?collection=col10687
[872]http://cnx.org/content/m19221/latest/?collection=col10686
[873]http://cnx.org/content/m19451/latest/?collection=col10687
[874]http://cnx.org/content/m19190/latest/?collection=col10686
[875]http://cnx.org/content/m19210/latest/?collection=col10686
[876]http://cnx.org/content/m18290/latest/?collection=col10624
[877]http://cnx.org/content/m19447/latest/?collection=col10687
[878]http://cnx.org/content/m19188/latest/?collection=col10686
[879]http://cnx.org/content/m18289/latest/?collection=col10624
[880]http://cnx.org/content/m19442/latest/?collection=col10687
[881]http://cnx.org/content/m19193/latest/?collection=col10686
[882]http://cnx.org/content/m18292/latest/?collection=col10624
[883]http://cnx.org/content/m19212/latest/?collection=col10686
[884]http://cnx.org/content/m19450/latest/?collection=col10687
[885]http://cnx.org/content/m19204/latest/?collection=col10686
[886]http://cnx.org/content/m19209/latest/?collection=col10686
[887]http://cnx.org/content/m19454/latest/?collection=col10687

[888]http://cnx.org/content/m18277/latest/?collection=col10624
[889]http://cnx.org/content/m18281/latest/?collection=col10624
[890]http://cnx.org/content/m19452/latest/?collection=col10687
[891]http://cnx.org/content/m19228/latest/?collection=col10686
[892]http://cnx.org/content/m19227/latest/?collection=col10686
[893]http://cnx.org/content/m19225/latest/?collection=col10686
[894]http://cnx.org/content/m18278/latest/?collection=col10624
[895]http://cnx.org/content/m18279/latest/?collection=col10624
[896]http://cnx.org/content/m19224/latest/?collection=col10686
[897]http://cnx.org/content/m19232/latest/?collection=col10686
[898]http://cnx.org/content/m18280/latest/?collection=col10624
[899]http://cnx.org/content/m19453/latest/?collection=col10687
[900]http://cnx.org/content/m19231/latest/?collection=col10686
[901]http://cnx.org/content/m19222/latest/?collection=col10686
[902]http://cnx.org/content/m19307/latest/?collection=col10687
[903]http://cnx.org/content/m18265/latest/?collection=col10624
[904]http://cnx.org/content/m18246/latest/?collection=col10624
[905]http://cnx.org/content/m19081/latest/?collection=col10686
[906]http://cnx.org/content/m19299/latest/?collection=col10687
[907]http://cnx.org/content/m19086/latest/?collection=col10686
[908]http://cnx.org/content/m18245/latest/?collection=col10624
[909]http://cnx.org/content/m19298/latest/?collection=col10687
[910]http://cnx.org/content/m19078/latest/?collection=col10686
[911]http://cnx.org/content/m19084/latest/?collection=col10686
[912]http://cnx.org/content/m18268/latest/?collection=col10624
[913]http://cnx.org/content/m19079/latest/?collection=col10686
[914]http://cnx.org/content/m19315/latest/?collection=col10687
[915]http://cnx.org/content/m19086/latest/?collection=col10686
[916]http://cnx.org/content/m19313/latest/?collection=col10687
[917]http://cnx.org/content/m19092/latest/?collection=col10686
[918]http://cnx.org/content/m19311/latest/?collection=col10687
[919]http://cnx.org/content/m19094/latest/?collection=col10686
[920]http://cnx.org/content/m18247/latest/?collection=col10624
[921]http://cnx.org/content/m19303/latest/?collection=col10687
[922]http://cnx.org/content/m19083/latest/?collection=col10686
[923]http://cnx.org/content/m19088/latest/?collection=col10686
[924]http://cnx.org/content/m19095/latest/?collection=col10686
[925]http://cnx.org/content/m19305/latest/?collection=col10687
[926]http://cnx.org/content/m18249/latest/?collection=col10624
[927]http://cnx.org/content/m19306/latest/?collection=col10687
[928]http://cnx.org/content/m19082/latest/?collection=col10686
[929]http://cnx.org/content/m19089/latest/?collection=col10686
[930]http://cnx.org/content/m18270/latest/?collection=col10624
[931]http://cnx.org/content/m19093/latest/?collection=col10686
[932]http://cnx.org/content/m19495/latest/?collection=col10687
[933]http://cnx.org/content/m19076/latest/?collection=col10624
[934]http://cnx.org/content/m19285/latest/?collection=col10686
[935]http://cnx.org/content/m19490/latest/?collection=col10687
[936]http://cnx.org/content/m19284/latest/?collection=col10686
[937]http://cnx.org/content/m19074/latest/?collection=col10624
[938]http://cnx.org/content/m19491/latest/?collection=col10687
[939]http://cnx.org/content/m19280/latest/?collection=col10686
[940]http://cnx.org/content/m19494/latest/?collection=col10687
[941]http://cnx.org/content/m19282/latest/?collection=col10686
[942]http://cnx.org/content/m19075/latest/?collection=col10624
[943]http://cnx.org/content/m19492/latest/?collection=col10687
[944]http://cnx.org/content/m19281/latest/?collection=col10686
[945]http://cnx.org/content/m19493/latest/?collection=col10687
[946]http://cnx.org/content/m19283/latest/?collection=col10686
[947]http://cnx.org/content/m19236/latest/?collection=col10686
[948]http://cnx.org/content/m19463/latest/?collection=col10687

[949]http://cnx.org/content/m19234/latest/?collection=col10686
[950]http://cnx.org/content/m19073/latest/?collection=col10624
[951]http://cnx.org/content/m19237/latest/?collection=col10686
[952]http://cnx.org/content/m19461/latest/?collection=col10687
[953]http://cnx.org/content/m19233/latest/?collection=col10686
[954]http://cnx.org/content/m19464/latest/?collection=col10687
[955]http://cnx.org/content/m19235/latest/?collection=col10686
[956]http://cnx.org/content/m19072/latest/?collection=col10624
[957]http://cnx.org/content/m19462/latest/?collection=col10687
[958]http://cnx.org/content/m19241/latest/?collection=col10686
[959]http://cnx.org/content/m19071/latest/?collection=col10624
[960]http://cnx.org/content/m19460/latest/?collection=col10687
[961]http://cnx.org/content/m19240/latest/?collection=col10686
[962]http://cnx.org/content/m19238/latest/?collection=col10686

Chapter 1

Functions

1.1 Introduction[1]

This is the most important unit in the year, because it introduces many of the major "themes" that will run through the entire class. These themes include:

- What is a function?
- How are functions used to model things in the real world?
- What does it mean for two functions to be "equal"?
- How does a functional description $f(x) = 3x$ relate to a graph?

More detailed topics include the vertical line test (the "rule of consistency"), the "dependent" and "independent" variables, domain and range, composite functions, and inverse functions.

Also included in this unit is a quick review of graphing lines. It is assumed that students are mostly familiar with this topic from Algebra I.

1.2 The Function Game[2]

This game is an introduction to the idea of a function.

Begin by breaking the students into groups of three. In each group, one student is designated as the **leader** and another as the **recorder**. (Do this quickly and arbitrarily: "The shortest person is the leader and the tallest is the recorder" or some such. Assure them that the roles will rotate.) Go over the instructions (which are in the student packet): walk through a sample session, using the function "add five," to make sure they understand who does what and what gets written down. In particular, make sure they understand how "add five" can be represented as "$x + 5$": that is, that x is being used to designate the number that comes **in**. You also want to mention **domain** in particular, and the idea that if you are doing $\frac{1}{x}$ and someone gives you a 0, it is "not in your domain."

Then they can start. Your job is to circle around, keeping them on task, and helping students who are stuck by giving hints: "Do you notice anything in common about all the numbers you've gotten back?" or "Why are all negative numbers outside the domain?" or even "Try a 2 and see what happens." Also, at 10 or 15 minute intervals, instruct them to switch roles.

Toward the end of the class period, interrupt briefly to talk about the word **function**. What is the leader representing? He is not a number. He is not a variable. He is a **process** that turns one number into another. That's all a function is—a mechanical process that takes one number in, and spits a different number back

[1] This content is available online at <http://cnx.org/content/m19335/1.2/>.
[2] This content is available online at <http://cnx.org/content/m19342/1.2/>.

out. Use the analogy of a little machine (you can draw it) with an input and an output. Functions are going to be the main focus of the entire year in Algebra II.

If some groups don't finish all the problems, that's OK: as long as they did enough to get the idea. If a group finishes early, tell them to start making up their own functions—challenge them to stump you!

Homework:

"Homework: The Function Game"

1.3 "The Real World"[3]

Begin by going over the homework. There are two key points to bring out, which did not come up in class yesterday.

1. If two functions always give the same answer, we say they are equal. This just cannot be stressed enough. Put this on the board:

$3(x-5)=3x-15$

They've all seen this. Most of them know it is the "distributive property." But do they know what it **means**? Explain it very carefully. $3(x-5)$ is a function that says "subtract 5, then multiply by 3." $3(x-15)$ is a function that says "multiply by 3, then subtract 15." They are very different processes! But we say they are "equal" because no matter what number you plug in, they yield the same answer. So in the function game, there is no possible question you could ask that would tell you if the person is doing "subtract 5, then multiply by 3" or "multiply by 3, then subtract 5." Have the class give you a few numbers, and show how it works—for negative numbers, fractions, zero, **anything**.

2. The last problem on the homework brings up what I call the "rule of consistency." It is perfectly OK for a function to give the same answer for different questions. (For instance, x^2 turns both 3 and -3 into 9.) But it is **not** OK to give different answers to the **same** question. That is, if a function turns a 3 into a 9 once, then it will **always** turn a 3 into a 9.

Once those two points are very clearly made, and all questions answered, you move on to today's work. Note that there is no "in-class assignment" in the student book: this is a day for interacting with the students.

Remind them that a function is simply a process—any process—that takes one number in, and spits a different number out. Then explain that functions are so important because they model relationships in the real world, where one number **depends** on a different number. Give them a few examples like the following—no math, just a verbal assertion that one number depends on another:

- The number of toes in class depends on the number of feet in class.
- Which, in turn, depends on the number of people in class.
- The number of points you make in basketball depends on how many baskets you make.
- The amount I pay at the pump depends on the price of gas.

Have them brainstorm in pairs (for 1-2 minutes at most) to come up with as many other examples as they can. Since this is the first "brainstorming" exercise in class, you may want to take a moment to explain the concept. **The goal of brainstorming is quantity, not quality**. The object is to come up with as many examples as you can, no matter how silly. But although they may be silly, they must in this case be **valid**. "The color of your shirt depends on your mood" is not a function, because neither one is a number. "The number of phones in the class depends on the number of computers" is not valid, because it doesn't. By the end of a few minutes of brainstorming and a bit more talk from you, they should be able to see how easy it is to find numbers that depend on other numbers.

Then—after the intuitive stuff—introduce formal functional notation. Suppose you get two points per basket. If we let p represent the number of points, and b represent the number of baskets, then $p(b)=2(b)$. If we say $p(c)=2c$ that is not a different function, because they are both ways of expressing the idea that "p doubles whatever you give it" (relate to the function game). So if you give it a 6, you get $12: p(6)=12$. If you give it a duck, you get two ducks: $p(\text{duck})=2\text{ducks}$. It doubles whatever you give it.

[3] This content is available online at <http://cnx.org/content/m19331/1.2/>.

Key points to stress:

- This notation, $p(b)$, does **not** indicate that p is being multiplied by b. It means that p **depends on** b, or (to express the same thing a different way), p **is a function of** b.
- You can plug any numbers you want into this formula. $p(6) = 12$ meaning that if you get 6 baskets, you make 12 points. $p\left(2\frac{1}{2}\right) = 5$ is valid mathematically, but in the "real world" you can't make $2\frac{1}{2}$ baskets. Remind them that p is a **function**—a process—"double whatever you are given."
- Also introduce at this point the terminology of the **dependent** and **independent** variables.
- Stress clearly defining variables: not "b is baskets" but "b is the number of baskets you make." ("Baskets" is not a **number**.)
- Finally, talk about how we can use this functional notation to ask questions. The question "How many points do I get if I make 4 baskets?" is expressed as "What is $p(4)$?" The question "How many baskets do I need in order to get 50 points?" is expressed as "$p(b) = 50$, solve for b."

For the rest of class—whether it is five minutes or twenty—the class should be making up their own functions. The pattern is this:

1. Think of a situation where one number depends on another. ("Number of toes depends on number of feet.")
2. Clearly label the variables. (t=number of toes, f=number of feet.)
3. Write the function that shows how the **dependent** variable depends on the **independent** variable ($t(f) = 5f$.)
4. Choose an example number to plug in. (If there are 6 feet, $t(f) = 5(6) = 30$. 30 toes.)

Encourage them to think of problems where the relationship is a bit more complicated than a simple multiplication. ("The area of a circle depends on its radius, $A(r) = \pi r^2$").
Homework:
"Homework: Functions in the Real World."

When going over this homework the next day, one question that is almost **sure** to come up is #4g: $f(f(x))$. Of course, I'm building up to the idea of composite functions, but there is no need to mention that at this point. Just remind them that $f(\text{anything}) = \text{anything}^2 + 2\text{anything} + 1$. So the answer to (e) is $f(\text{spaghetti}) = \text{spaghetti}^2 + 2\text{spaghetti} + 1$. And the answer to this one is $f(f(x)) = f(x)^2 + 2f(x) + 1$. Of course, this can (and should) be simplified, but the point right now is to stress that idea that you can plug anything you want in there.

1.4 Algebraic Generalizations[4]

This is the real fun, for me.

Start by telling the students "Pick a number. Add three. Subtract the number you started with. You are left with...three!" OK, no great shock and surprise. But let's use algebra to express what we have just discovered. $x + 3 - x = 3$. The key is recognizing what that sentence mean. x can be any number. So when we write $x + 3 - x = 3$ we are indeed asserting that if you take **any number**, add three and then subtract the number, you get three in the end.

Here's a harder one. Pick a number, add three, multiply by four, subtract twelve, divide by the number you started with. Everyone started with different numbers, but everyone has 4 in the end. Ask the students to find a generalization to represent **that**, and see if they can work their way to $\frac{4(x+3)-12}{x} = 4$. See also if they can guess what number this trick will **not** work with.

Now, have them work on the in-class assignment "Algebraic Generalizations," in groups of three.. Most of the class period should be spent on this. This is hard!!! After the first couple of problems (which very directly echo what you already did in class), most groups will need a lot of help.

[4]This content is available online at <http://cnx.org/content/m19332/1.3/>.

Here are some of the answers I'm looking for—I include this to make sure that the purpose of the assignment is clear to teachers.

- In #3, the object is to get to $2^{x+1} = 2 \bullet 2^x$ (or, equivalently, $2^x = 2 \bullet 2^{x-1}$. Talk through this very slowly with individual groups. If you wanted to get from 2^{10} to 2^{11} , what would you do? And to get from 2^{53} to 2^{54} ? And what about 2^{99} to 2^{100} ? Can you say **in English** what we're saying, in general? Now, can you say that in math? etc... The real goal is to get them to see how, once you have written $2^{x+1} = 2 \bullet 2^x$, you have said in one statement that 2^8 is twice 2^7, and also that 2^5 is twice 2^4, and also that 2^{11} is twice 2^{10}, and so on. It is a "generalization" because it is one statement that represents many separate facts.
- In #4, the object is to get to $x^a x^b = x^{a+b}$. Again, it will take a lot of hand-holding. It isn't important for them to do it entirely on their own. It is critically important that, by the time they are done, they see how those numbers lead to that generalization; and how that generalization leads to those numbers.
- If all that works, they should be able to do #6 and come up with something like $(x - -1)(x+1) = x^2 - 1$ pretty much on their own. (Or, of course, $x^2 = (x-1)(x+1) + 1$, which is a bit more unusual-looking but just as good.)

Homework:
"Homework: Algebraic Generalizations"

1.5 Graphing[5]

Make sure, on the homework, that they reached something like $(x-a)(x+a) = x^2 - a^2$. Of course, someone may have used completely different letters; and someone else may have said that $(x-a)(x+a) + a^2 = x^2$. Point out that these are just as good—they look different, but they say the same thing. Finally, show them how they could have arrived at $(x-a)(x+a) = x^2 - a^2$ through FOIL.

Once all questions are satisfied, on to graphing. This may actually be two days worth of material. Don't rush it! And once again, no in-class assignment, but a lot of interaction.

Start by putting the following points on the board: (6,5)(3,6)(2,9)(5,7)(1,10)(4,8). Ask for a general description—if these represent the number of pushups you do each day, what's the trend? Then graph them, and you can see a definite downward trend with an odd spike in the middle. Moral: we can see things in **shapes** that we can't see in **numbers**.

Now, let's jump to the idea of graphing functions. Draw a U-shape, the equation $f(x) = x^2$, and say "This drawing is the graph of that function—but what does that **mean**? What does this drawing actually have to do with that function?" Get to the point where the following ideas have come out. Every time you "do" the function, one number goes in and another comes out. When we graph it, the "in" function is always x, and the "out" function is always y—in other words, **every time we graph a function, we are always graphing** $y = f(x)$. To put it another way, we are graphing all the points that have this particular relationship to each other. (Take as much time as you need on this point.)

Have the students graph $|x|$ (individually at their desks) by plotting points. This is not intended to teach them about absolute value, but to reinforce the ideas I just made about what it means to graph a function.

What can we tell by looking at a graph? Draw the graphs of x^2 and x^3 on the board. Talk about the things we can tell about these two **functions** by looking at the **graphs**. They both have one zero; they are similar on the right, except that x^3 rises faster; they are completely different on the left; they both have unlimited domains, but only one has an unlimited range. (Make sure to connect this back to "domain" and "range" from the function game!)

Now, draw this on the board.

[5]This content is available online at <http://cnx.org/content/m19334/1.3/>.

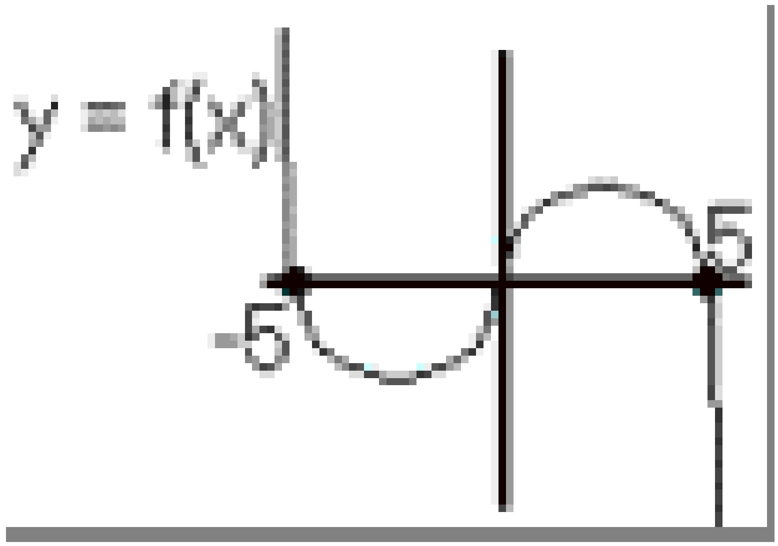

Figure 1.1

Time for another...brainstorming exercise! (Remind them that the object is quantity, not quality!) Each pair of students has to list as many things as it can tell about this function $f(x)$ by looking at the graph. Key points I want to bring out are:

- The three zeros. (Talk about the word "zeros.")
- Places where the function is negative and places where it is positive.
- Places where it is increasing and places where it is decreasing. (Talk about slope!)
- What happens for very low and very high values of x.
- For the experts, odd symmetry: $f(-x) = -f(x)$.

Talk more about domain and range. This is unrestricted in both. We saw that x^2 has an unrestricted domain, but a restricted range. Why would any function have a limited domain? Generally, it is because of the two "thou shalt not" rules: **thou shalt not divide by zero,** and **thou shalt not take the square root of a negative number.** So, consider the following three statements.

1. You can't take the square root of a negative number.
2. The domain of the function y=$\sqrt{x}$ is x≥0.
3. The graph of y=$\sqrt{x}$ starts at the y-axis, and goes to the right; it doesn't go to the left. (Draw it.)

Give them all three of these statements, and see if they can see that **they are all three saying the same thing.** Then see if they can generate three equivalent versions of "You can't divide by zero." ("The domain of the function y=$1/x$ is all numbers except 0," and "The graph of y=$1/x$ never touches the y-axis.")

Remind them of the "rule of consistency" that we discussed earlier: a function can never take one **input** and generate two different **outputs**. Ask them to discuss in pairs, for one minute, how this rule manifests on a graph. Then have class discussion until you have reached the vertical line test.

Come back to the question we started with: why do we graph things? As we demonstrated earlier, graphs enable us to see things visually that are very hard to see in numbers. Draw several different jaggy, shaky graphs, and suggest that they represent the price of gas—ask for verbal descriptions of what each one tells us. Talk again about domain, range, positive, negative.

Throw something straight up into the air and catch it. Tell them there is a function $h(t)$ that represents the height of that object, as a function of time. (Make sure they get this.) Give them one minute to sketch the graph of that function. Then show them that it is an upside-down parabola (you don't need to use that word). Emphasize that this does **not** mean the object traveled in an arc shape: it traveled straight up and down. The horizontal axis is **time** and the vertical axis is height.

If you have a bit of extra time, explain how to generate graphs (and set the window) on the calculator.

Homework:

"Homework: Graphing"

1.6 Permutations[6]

The end of the "Graphing" homework sets this topic up.

Have one person in the class "be" x^2. He is allowed to use a calculator; so you can, for instance, hand him the number 1.7 and he will square it, producing the point (1.7,2.89).

Another person is x^2+1. He is *not* allowed to use a calculator, but he *is* allowed to talk to the first person, who is. So if you hand him 1.7, he asks the first person, who says 2.9, and then he comes back with a 3.9. Make sure everyone understands what we have just learned: the graph of x^2+1 contains the point (1.7,3.9). Do a few points this way.

Another person is $(x+1)^2$ with the same rules. So if you give him a 1.7 he hands a 2.7 to the calculator person. Make sure everyone understands how this process gives us the point (1.7,7.3).

Talk about the fact that the first graph is a *vertical* permutation: it messed with the y-values that came out of the function. It's easy to understand what it did. It added 1 to every y-value, so the function went up 1.

The second graph is a *horizontal* permutation: it messed with the x-values that went into the function. It's harder to see what that did: why did $(x+1)^2$ move to the left? Ask them to explain that.

Now hand them the worksheet "Horizontal and Vertical Permutations I." Hand one to each person—they will start in class, but probably finish in the homework. It's on the long side.

The next day, talk it all through very carefully. Key points to bring out:

1. What does $f(x)+2$ mean? It means first plug a number into $f(x)$, and then add 2.
2. And what does that do to the graph? It means every y-value is two higher than it used to be, so the graph moves up by 2.
3. What does $f(x+2)$ mean? It means first add 2, then plug a number into $f(x)$.
4. And what does that do to the graph? It means that when $x=3$ you have the same y-value that the old graph had when $x=5$. So your new graph is to the *left* of the old one.
5. What does all that have to do with our rock? This should be a long-ish conversation by itself. The vertical and horizontal permutations represent very different types of changes in the life of our rock. Suggest a different scenario, such as our old standard, the number of candy bars in the room as a function of the number of students, $c(s)$. What scenario would $c(s)+3$ represent? How about $c(s+3)$?

If you haven't already done so, introduce graphing on the calculator, including how to properly set the window. It only takes 5-10 minutes, but is necessary for the homework.

Now, put the graph of $y=x^2$ on the board. We saw what $(x+1)^2$ and x^2+1 looked like yesterday. What do you think $-x^2$ would look like? How about $(x+2)^2-3$?

At some point, during the first or second day, you can come back to the idea of domain. What is the domain of $y=\sqrt{x+3}$? See if they can see the answer both numerically (you can plug in $x=-3$ but not $x=-4$ and graphically (the graph of $y=\sqrt{x}$ moved three spaces to the left, and its domain moved too).

Homework:

"Horizontal and Vertical Permutations II"

[6]This content is available online at <http://cnx.org/content/m19339/1.3/>.

1.7 Test Preparation[7]

Follow up carefully on the homework from the day before. In #8, make sure they understand that losing money is **negative** profit. But mostly talk about #9. Make sure they understand how, and why, each modification of the original function changed the graph. When we make a statement like "Adding two to a function moves the graph **up** by 2" this is not a new rule to be memorized: it is a common-sense result of the basic idea of graphing a function, and should be understood as such.

Ask what they think $--(x+1)^2--3$ would look like, and then show them how it combines three of the modifications. (The $+1$ moves it to the left, the -3 moves it down, and the $-$ in front turns it upside-down.)

Talk about the fact that you can do those generalizations to any function, eg $--|x+1|--3$ (recalling that they graphed absolute value yesterday in class).

Draw some random squiggly $f(x)$ on the board, and have them draw the graph of $f(x)+2$. Then, see if they can do $f(x+2)$.

Go back to the idea of algebraic generalizations. We talked about what it means for two functions to be "equal"—what does that look like, graphically? They should be able to see that it means the two functions have exactly the same graph. But this is an opportunity for you to come back to the main themes. If two functions are equal, they turn every x into the same y. So their graphs are the same because a graph is all the (x,y) pairs a function can generate!

I also like to mention at this point that we actually use the $=$ sign to mean some pretty different things. When we say $x+3=5$ we are asking **for what value of x is this true?** Whereas, when we say $2(x-7)=2x-14$ we are asserting that **this is true for all values of** x. Mention, or ask them to find, statements of equality that are not true for **any** value of x (eg $x=x+1$).

Now, at this point, you hand out the "sample test." Tell them this test was actually used for a past class; and although the test you give will be different, this test is a good way of reviewing. I have found this technique—handing out a real test from a previous year, as a review—to be tremendously powerful. But I have to say a word here about how to use it. **Sometimes** I say "Work on it in class if you have time, glance it over tonight if it helps you study; the test is tomorrow." And **sometimes** I say "This is the homework. Tonight, do the sample test, and also look over all the materials we have done so far. Tomorrow, we will go over the sample test and any questions you have on the material, in preparation for the test, which will be the day after tomorrow." It all depends on timing, and on how prepared you think the class is.

Homework:

"Sample Test: Functions I"

I should say a word about #2 here, just to be clear about what I'm looking for. We have a function $c(s)$. In part (b) I supply $s=20$ and ask for c: so the question in function notation is $c(20)$. (Then you plug a 20 into the formula and go from there.) In part (c) I supply $c=35$ and ask for $c(20)$: so this question in function notation is $c(s)=35$. (Then you set the formula equal to 35 and solve.) Students have a lot of difficulty with this asymmetry.

1.7.1 Now give a test of your own on functions I

You may use something very similar to my test, except changing numbers around. Or you may do something quite different.

I will tell you, as a matter of personal bias, that I feel very strongly about question #7. Many students will do quite poorly on this question. For instance, they may give you two variables that do not in fact depend on each other (number of guitarists and number of drummers). Or they may give you variables that are in fact constants (let n equal the number of notes in an octave). The answers don't have to be complicated, although sometimes they are—sometimes a very simple answer gets full credit. ("CDs cost $12 apiece, I spend d dollars on c CDs. $d(c)=12c$" is a full credit answer to parts a-c.) But they have to show that they can clearly articulate what the variables are, and how they depend on each other. If they cannot, spend the time to explain why it is wrong, and work them through correct answers. In my opinion, this

[7]This content is available online at <http://cnx.org/content/m19340/1.2/>.

skill is the best measure of whether someone really understands what a function is. And it is a skill like any other, in the sense that it develops over time and practice.

(Also, I don't believe in surprises. Tell them in advance that this question will definitely be on the test—the only thing that will change is the topic.)

1.8 Lines[8]

This is largely review: if there is one thing the students **do** remember from Algebra I, it's that $y = mx + b$ and m is the slope and b is the y-intercept. However, we're going to view this from the viewpoint of **linear functions.**

Start by giving an example like the following: I have 100 markers in my desk. Every day, I lose 3 markers. Talk about the fact that you can write a function $m(d)$ that represents the number of markers I have as a function of day. It is a **linear** function because it **changes by the same amount every day**. If I lose three markers one day and four the next, there is still a function $m(d)$, but it is no longer a **linear** function. (If this is done right, it sets the stage for exponential functions later: linear functions **add** the same amount every day, exponential functions **multiply** by the same amount every day. But I wouldn't mention that yet.)

So, given that it changes by the same amount every day, what do you need to know? Just two things: how much it changes every day (-3), and where it started (100). These are the slope and the y-intercept, respectively. So we can say $y = -3x + 100$ but I actually prefer to write the b first: $y = 100 - 3x$. This reads very naturally as "start with 100, and then subtract 3, x times."

Hammer this point home: a linear function is one that adds the same amount every time. Other examples are: I started with $100 and make $5.50 each hour. (Money as a function of time.) I start on a 40' roof and start piling on bricks that are $\frac{1}{3}$ ' each. (Height as a function of number of bricks.)

Then talk more about slope—that slippery concept that doesn't tell you **how high** the function is at all, but just **how fast it's going up**. With a few quick drawings on the board, show how you can look at a line and guesstimate its slope: positive if it's going up, negative if it's going down, zero for horizontal. You can't necessarily tell the difference between a slope of 3 and a slope of 5, but you can immediately see the difference between 3 and $\frac{1}{2}$. Emphasize that when we say "going up" and "going down" we always mean **as you go from left to right**: this is a very common source of errors.

Talk about the strict definition of slope. Actually, I always give two definitions. One is: every time x increases by 1, y increases by the slope. (Again: if the slope is negative, y decreases.) The other is: for **any two points** on the line, the slope is $\frac{\Delta x}{\Delta y}$ ("rise over run"). Show that this ratio is the same whether you choose two points that are close, or two points that are far apart. Emphasize that this is only true for lines.

Finally, why is it that in $y = mx + b$, the b is the y-intercept? Because the y-intercept is, by definition, the value of y when $x = 0$. If you plug $x = 0$ into $y = mx + b$ you get $y = b$.

All this may take all day, or more than one day. Or, it may go very quickly, since so much of it is review. When you're done, have them work the in-class assignment "Lines" in groups.

Homework:

"Homework: Graphing Lines"

1.9 Composite Functions[9]

OK, it's getting hard again: this one may take a couple of days. We start with discussion.

There are a number of ways to look at composite functions. It's important to be able to use **all** of these ways, and to see how they relate.

1. All the way back to the function game. Let one student be the function $4x + 6$, and another student be the function $x/2$. You give a number to the first student, who spits out a number at the second

[8] This content is available online at <http://cnx.org/content/m19337/1.2/>.
[9] This content is available online at <http://cnx.org/content/m19333/1.2/>.

student, who spits out a number back at you. For instance, if you give the first student a 3, his output is an 18; the second student takes this 18 and comes out with a 9. Do this for a while until everyone has the hang of it. See if anyone realizes that what's going on is, in the end, the function $2x + 3$ is being done to your number.

2. Now, talk about a factory. One box turns garbage into gloop; the next box turns gloop into shlop; the final box turns shlop into food. Each box can be represented by a function that says "If this much goes in, that much goes out." The entire factory is a gigantic composite function, where the output of each box is the input of the next, and the composite function says "If this much garbage goes in, this much food goes out." (Draw it!)
3. In general, composite functions come up with **this variable depends on that variable, which in turn depends on the other variable**. The amount of taxes you pay depends on the amount of money you make, which in turn depends on the number of hours you work. Have them come up with a few examples. Be very careful to distinguish composite functions from multivariate functions, **e.g**. the number of kids in the class depends on the number of boys, and the number of girls. That is not a function, because those two variables don't depend on each other.
4. Finally, there is the formalism, $f(g(x))$. Remind them that this is mechanical. If $g(x) = 4x + 6$ and $f(x) = x/2$, then what is $f(g(x))$? Well, $f(\text{anything}) = \text{anything}/2$. So $f(g(x)) = g(x)/2$, which is $(4x + 6)/2$ or $2x + 3$. Note that this is completely different from $g(f(x))$! Take a moment to connect this mechanical process with the idea of a composite function that you have already discussed.

Now, have them work through the in-class assignment on "Composite Functions" in groups. Make sure they do all right on #4.

#6 is a build-up to inverse functions, although you don't need to mention that. If anyone asks for help, help them see that if $h(x) = x - 5$, then $h(\text{anything}) = \text{anything} - 5$, so $h(i(x)) = i(x) - 5$. So $i(x) - 5 = x$, and we can solve this to find $i(x) = x + 5$.

Homework:

"Homework: Composite Functions"

1.10 Inverse Functions[10]

This is definitely two days, possibly three.

Just as with composite functions, it is useful to look at this three different ways: in terms of the function game, in terms of real world application, and in terms of the formalism.

1. Ask a student to triple every number you give him and then add 5. Do a few numbers. Then ask another student to **reverse what the first student is doing**. This is very easy. You give the first student a 2, and he gives you an 11. Then you give the second student an 11, and he gives you a 2. Do this a few times until everyone is comfortable with what is going on. Then ask what function the **second** student is doing. With a little time, everyone should be able to figure this out—he is reversing what the first student did, so he is subtracting five, then dividing by 3. These two students are "inverses" of each other—they will always reverse what the other one does.

2. Give a few easy functions where people can figure out the inverse. The inverse of $x + 2$ is $x - 2$ (and vice-versa: it is always symmetrical). The inverse of x^3 is $\sqrt[3]{x}$

The key thing to stress is how you **test an inverse function**. You try a number. For instance...

$10 \rightarrow x + 2 \rightarrow 12 \rightarrow x - 2 \rightarrow 10$
$-5 \rightarrow x + 2 \rightarrow -3 \rightarrow x - 2 \rightarrow -5$

[10]This content is available online at <http://cnx.org/content/m19336/1.2/>.

Table 1.1

The point is that you take any number and put it into the first function; put the answer in the second function, and you should get back to your original number. Testing inverses in this way is more important than finding them, because it shows that you know what an inverse function **means**.

3. Ask for the inverse of x^2. Trick question: it doesn't have any! Why not? Because x^2 turns 3 into 9, and it also turns –3 into 9. It's allowed to do that, it's still a function. But an inverse would therefore have to turn 9 into both 3 and –3, and a function is **not** allowed to do that—rule of consistency. So x^2 is a function with no inverse. See if the class can come up with others. (Some include $y = |x|$ and $y = 3$.)
4. Now ask them for the inverse of $10 - x$. They will guess $10 + x$ or $x - 10$; make sure they test! They have to discover for themselves that these don't work. The answer is $10 - x$; it is its own inverse. (It turns 7 into 3, and 3 into 7.) Ask for other functions that are their own inverses, see if they can think of any. (Other examples include $y = x$, $y = -x$, $y = 20/x$.)
5. In practice, inverse functions are used to go backwards, as you might expect. If we have a function that tells us "If you work this many hours, you will get this much money," the inverse function tells us "If you want to make this much money, you have to work this many hours." It reverses the x and the y, the dependent and independent variables. Have the class come up with a couple of examples.
6. Formally, an inverse function is written $f^{-1}(x)$. This does not mean it is an exponent, it is just the way you write "inverse function." The strict definition is that $f\left(f^{-1}(x)\right) = x$. This definition utilizes a composite function! It says that if x goes into the inverse function, and then the original function, what comes out is…x. This is a hard concept that requires some talking through.

OK, at this point, you have them start working on the in-class exercise "Inverse Functions." Note that you have not yet given them a way of **finding** inverse functions, except by noodling around! Let them noodle. Even for something like $y = \frac{2x+3}{7}$, they should be able to get there, with a bit of hand-holding, by reversing the steps: first multiply by 7, then subtract 3, then divide by 2. If they ask about #11, make sure they try a few things (such as $\sqrt[x]{x}$) and **test them**—they will discover they don't work. Explain that, in fact, we have no inverse function of 2^x right now, so we're going to make one up later in the year and call it a "logarithm." They can then leave this one blank.

After they have finished noodling their way through the most of the exercises, interrupt the class and say "Now, I'm going to give you a formal method of finding inverse functions—you will need this for the homework." The formal way is: first, reverse the x and the y, then solve for y. For instance, from $y = \frac{2x+3}{7}$ we first write $x = \frac{2y+3}{7}$. Then solve for y to get $y = \frac{7x-2}{3}$.

Homework:

"Homework: Inverse Functions"

But wait! We're not done with this topic!

What happens the next day is, they come in with questions. Whatever else they did or didn't get, they got stuck on #10. (If they got stuck on #9, point out that it is the same as #2; make sure they understand why.) #10 is hard because they cannot figure out how to solve for y. This brings us, not to a new conceptual point, but to a very important algebraic trick, which we are going to learn by doing a TAPPS exercise.

TAPPS (Thinking Aloud Pair Problem Solving) is a powerful learning tool, and here's how it goes. The students are broken into pairs.

One person in each pair is the **teacher**. His job is to walk through the following solution, step by step, explaining it. For each step, explain two things. Why am I **allowed** to do that, and why did I **want** to do that? Your explanation should make perfect sense to a normal Algebra I student. You should never skip steps—go line, by line, by line, explaining each one.

The other person is the **student.** He also has two jobs. First, whenever the teacher says something that is not perfectly clear, stop him! Even if you understand it, say "Wait, that didn't make perfect sense." Keep pushing until the explanation is completely bullet-proof. Second, keep the teacher talking. If he pauses to think, say "Keep talking. What are you thinking?" The teacher should "think out loud" until he comes up with something.

If the students are stuck on a line, they should raise their hands and ask you.

Take the time to carefully explain the process—we will do other TAPPS exercises. And one more thing—warn them that after everyone has completed the exercise, you will be calling on individuals to explain tricky steps. **You will not call for volunteers.** After they are done, everyone in the class should be able to answer any question about anything in this derivation. So you will just pick people and ask them questions like "Why did I do that?"

After they are done, call on individuals and ask questions like "Why did we subtract $2xy$ from both sides?" and "How did we get $y(1-2x)$?" Point out the general strategy is only two steps: get all the y things on one side, then pull out the y. Check their answer to the question at the end.

1.10.1 Time for another test!

Once again, there is the sample test—you will probably want to assign it as a homework, and tell them to do that and also study everything since the last test. The next day, go over the homework and any questions. Then give the test.

Congratulations, you're through with your first unit! If things went well, you have laid the groundwork for the entire year. Onward and upward from here!

Chapter 2

Inequalities and Absolute Values

2.1 Introduction[1]

I like to have this unit early in the year, because it introduces another of my main themes: some of these problems **require you to think**. There is almost no "mechanical" way to solve them. I want to set up that expectation early.

Of course, this is really two very different topics, and both of them contain an element of review as well as some new material. But by the time we get to combining them in problems like $2|3x+4|<7$, it is new to everybody. Those problems are harder than you might suppose!

2.2 Inequalities[2]

This is pretty easy, isn't it? You can just start by having them work through "Inequalities" with no preamble at all.

After they've worked on it for a while, you might want to interrupt the class to talk about it for a while. #1 is obviously an attempt to get at the old "you have to reverse the inequality when you multiply or divide by a negative number" thing. But stress as loudly as you can, the idea that they **shouldn't just take anyone's word for it**. The question we're trying to get at is, **why** do you have to switch the inequality, then and only then? I usually illustrate this point by drawing a number line and showing how, on the left of the zero (in "negative land") the numbers are going **backward** (an observation that every second grader notices, but that they have forgotten by high school). So you can visually show how $2<3$ becomes $--2>-3$ when you move it to the other side.

Also, tell them how important it is to distinguish carefully between ANDs and ORs—this will become a major issue later. I have two pet peeves on this topic.

One pet peeve is people who memorize a facile rule (such as "less-than problems become AND and greater than problems become OR") without having the slightest idea what they are doing. I go out of my way to create problems that frustrate such rules (which isn't hard to do). They have to see and understand what these conjunctions **mean**.

The other pet peeve is $x> \pm 4$. This is, for all intents and purposes, meaningless. I want them to realize that on #9, and then I warn them that I will always take off points if they answer any question this way. (This comes up in the context of $|x|>4$ and is a good example of how wrong you go if you answer mechanically instead of thinking.) x can **equal** ± 4, if you know what that means (shorthand for $x=4$ or $x=-4$), but it cannot be **greater than**, or **less than**, ±anything.

Homework:
"Homework: Inequalities"

[1] This content is available online at <http://cnx.org/content/m19432/1.3/>.
[2] This content is available online at <http://cnx.org/content/m19430/1.2/>.

2.3 Inequality Word Problems[3]

The difficulty with this is that there is really nothing to say about it at all: it's just something I want them to see. So there is this homework assignment, which you can give at any time, before or after anything else. Stick it in when you find yourself, at the end of a day, not quite ready to give out the next homework, or something like that.

Homework:

"Inequality Word Problems."

In one problem, they have to **make up** an inequality word problem. You might think that, just because they have made up so many word problems and functions by now, they would knock this one down easily—but it ain't so. I've had students who could create functions all day long (by this time) who could not create a good inequality to save their lives. They create scenarios like "I eat three bowls of cereal a day, how many do I eat in a week?" There is nothing **unequal** there. Just like everything else, this takes practice. But I do want them to see that inequality relationships are all around us.

2.4 Absolute Value Equations[4]

Now it gets tough. But once again, little or no preamble is needed: just have them start working on "Absolute Value Equations."

Here's the thing. Problems 1–9 really contain all the math: all the concepts they need to get. And, for the most part, they will get them right—although you need to check this before they go on any further.

But when it comes to the more complicated-**looking** problems in the second half of the assignment, they panic. They stop thinking, revert to rules, and start getting wrong answers. If they are diligently checking, they will realize that their answer to #12 doesn't work. But they may need you to point out that this is because it is analogous to #6 and has no answer.

So, around this time, I spend a lot of time insisting "Think, think, think!" The way to think it through is this. Once you have solved for the absolute value, go back to the kind of thinking you did in the first page. For instance, when you have $|x+3| = -1$, cover up the $x+3$ and ask yourself the question like this: "The absolute value of **something** is –1. What is the something?" The answer, of course, is "nothing." Think, and you will get it right. Plug and chug, and you will get it wrong.

OK, if $|x+3| = 7$ has two answers, and $|x+3| = 0$ has one, and $|x+3| = -4$ has none, then what about $|x - -2| = 2x - -10$? The answer is, **you don't know until you try**. You begin by splitting it the same way you did before: $x - -2 = 2x - -10$ or $x - -2 = - - (2x - -10)$. Find both answers. But then check them: **even if you did the math right, they may not work!** Don't tell them this up front, but make sure to discuss this with them toward the end of class, in the context of #13; they will need to know this for the homework.

Homework:

"Homework: Absolute Value Equations"

2.5 Absolute Value Inequalities[5]

They are going to work on the assignment "Absolute Value Inequalities" in class. You may want to begin by reminding them that they have already been solving absolute value inequalities. On the previous assignment they turned "the **absolute value of my number is less than 7**" into an inequality and solved it, by trying a bunch of numbers. These are no different. We are going to use the same sort of thinking process as before: confronted with $|3x-1| > 10$ we will say "OK, the absolute value of **something** is greater than 10. What

[3]This content is available online at <http://cnx.org/content/m19428/1.2/>.
[4]This content is available online at <http://cnx.org/content/m19426/1.2/>.
[5]This content is available online at <http://cnx.org/content/m19431/1.2/>.

could the **something** be? <think, think> OK, the **something** must be greater than 10 (like 11,12,13) or else less than –10 (like –11,-12,-13)." So we write $3x-1>10$ or $3x-1<-10$ and go from there.

"Gee, why are you making it so hard? My Algebra I teacher taught me that if it's greater than, just make it an "or" and if it's less than, just make it an "and.""

OK, let's try a slight variation: $|3x-1|>-10$. Now what? "OK, the absolute value of **something** is greater than –10. What could the **something** be? <think, think> OK, the **something** can be...anything!" The absolute value of anything is greater than –10, so any x-value will work!

"That's not what my Algebra I teacher taught me."

Fine, then, let's try that same problem your way. Then, let's test our answers, by plugging into the original inequality and see which answer works.

You get the idea? The whole point of this unit (to me) is to—very early in the year—establish a pattern that the only way to solve math problems is by thinking about them. This unit is great for that.

Homework:

"Homework: Absolute Value Inequalities"

2.6 Graphing Inequalities and Absolute Values[6]

This is a two-day topic, possibly three.

Start by putting the function $y=x^2--1$ on the board. Now, distinguish between two different kinds of questions.

1. Solve (or graph the solution of) $x^2--1<0$. This should remind the students of problems we did in the last unit, where we asked "For what x-values is this function negative?" The answer is $--1<x<1$; it could be graphed on a number line.
2. Graph $y<x^2--1$. This is a completely different sort of question: it is not asking "For what x-values is this true?" It is asking "For what $(x,\ y)$ pairs is this true?" The answer cannot live on a number line: it must be a shaded region on a two-dimensional graph. Which region? Well, for every point **on** this curve, the y-value is **equal** to x^2--1. So if you go **up** from there, then y is **greater than**...but if you go **down** from there, then y is **less than**... So you shade below it.

It is important to be able to solve both types of problems, but it is even more important, I think, to distinguish between them. If you answer the first type of question with a shaded area, or the second type on a number line, then you aren't just wrong—you're farther than wrong—you're not even thinking about what the question is asking. ("$2+2=5$" is wrong, but "$2+2=$ George Washington" is worse.)

With that behind you, get them started on the assignment "Graphing Inequalities and Absolute Values." They should get mostly or entirely finished in class, and they can finish it up and also do the homework that evening.

Homework:

"Homework: Graphing Inequalities and Absolute Values"

If they come in the next day asking about #4, by the way, just tell them to turn it into $y<-2|x|$ and then it is basically like the other ones.

Second day, no worksheet. After going over the homework, stressing the ways we permute graphs, warn them that here comes a problem that they cannot solve by permuting. Challenge them—first person with the correct shape is the winner, no calculators allowed—and then put on the board $y=x+|x|$. Give them a couple of minutes to plot points. Then let someone who got it right put it on the board—both the points, and the resultant shape.

Now, you point out that this shape is really a combination of two different lines: $y=2x$ on the right, and $y=0$ on the left. This odd two-part shape is predictable, **without** plotting points, if you understand absolute values in a different way. This is our lead-in to the **piecewise definition of the absolute value**:

[6]This content is available online at <http://cnx.org/content/m19433/1.2/>.

$$|x| = \{ \begin{array}{cc} x & x \geq 0 \\ -x & x < 0 \end{array}$$

This takes a whole lot of explaining: it is just one of those things that students find difficult. Here are a few ways to explain it (use all of them).

1. Just try numbers. If $x = 3$, then $|x| = 3$, so $|x| = x$. Same for $x = 4$, $x = 52$, and even $x = 0$. But if $x = -3$, then $|x| = 3$, so $|x| \neq x$ (they are not the same)! Instead, $|x| = -x$. Why, because $- - x$ in this case is $- - (-3)$ which is 3 which is indeed $|x|$.

But how can $|x| = -x$ **when** $|x|$ **is never negative**? Well, that brings us to…

2. Putting a – sign in front of a number does not make it negative: it **switches the sign**. It makes positive numbers negative, and negative numbers positive. So you can read that piecewise definition as "if x is negative, then the absolute value switches the sign."
3. Finally, come back to the graph of good old $y = |x|$. Point out that it is, indeed, the graph of $y = x$ on the right, and the graph of $y = -x$ on the left.

Now, how does all this relate to our original problem? When $x < 0$, we replace $|x|$ with $- - x$ so our function becomes $y = x - x = 0$. When $x \geq 0$, we replace $|x|$ with x so our function becomes $y = x + x = 2x$. That's why the graph came out the way it did.

Why is this important? It's an important way to understand what absolute value means. But it's also our first look at piecewise functions (one of the only looks we will get) so take a brief timeout to talk about why piecewise functions are so important. Throw an object into the air and let it drop, and talk about the function $h(t)$. We previously discussed this function only **during** the flight. But to get more general, you have to break it into three different functions: $h = 3$ before you throw it (assuming it was in your hand 3' above the ground), $h = 16 - t^2$ or something like that during the flight, and $h = 0$ after it hits the ground. Do a few more examples to get the idea across that piecewise functions come up all the time because conditions change all the time.

OK, back to our friend the absolute value. The students should now graph $y = \frac{x}{|x|}$ on their own (individually, not in groups), not by plotting points, but by breaking it down into three regions: $x < 0$, $x = 0$, and $x > 0$. (It is different in all three.) Get the right graph on the board.

Now, hopefully, you have at least 10-15 minutes left of class, because now comes the hardest thing of all. You're going to graph $|x| + |y| = 4$. Since x is under the absolute value, we have to break it into two pieces—the left and the right—just as we have been doing. Since y is under the absolute value, we also have to break it vertically. So what we wind up doing is looking at **each quadrant separately**. For instance, in the second quadrant, $x < 0$ (so we replace $|x|$ with $- - x$) and $y > 0$ (so we replace $|y|$ with y). So we have $y - x = 4$ which we then put into $y = mx + b$ format and graph, but **only in the second quadrant**. You do all four quadrants separately.

Explain this whole process—how to divide it up into the four quadrants, and how to rewrite the equation in the second quadrant. Then, set them going in groups to work the problem. Walk around and help. By the end of the class, most of them should have a diamond shape.

After they are all done, you may want to mention to them that this exact problem is worked out in the "Conceptual Explanations" at the very end of this chapter. So they can see it again, with explanations.

Homework:

Graph $|x| - 2|y| < 4$. This requires looking at each quadrant as a separate **inequality** and graphing them all in the appropriate places!

2.6.1 Time for another test!

Once again, there is the sample test—you will probably want to assign it as a homework, and tell them to do that and also study everything since the last test. The next day, go over the homework and any questions. Then give the test.

Chapter 3

Simultaneous Equations

3.1 Introduction to Simultaneous Equations[1]

Like all our topics so far, this unit reviews something the students covered in Algebra I—but it goes deeper.

Begin by having them work their way through the assignment "Distance, Rate, and Time" in pairs. Most of it should be pretty easy, including getting to the general relation $d = rt$. It should not take much time.

Until the last question, that is. Let them work on this for a while. Some will get all the way, some will not get very far at all. But after they've been at it for a while, tell them to stop working and pull them back to a classwide discussion. Show them how to set up $d = rt$ for each case, and make sure they understand. Maybe come up with another problem or two along the same lines, including one where the **times** are the same and the **distances** are different. (A train leaves Chicago and a train leaves New York, when do they crash…) Get them comfortable with **setting up** the two equations—we're not really focused on solving them.

Toward the end, see if they remember that there are three ways of solving these equations. Two of them, Substitution and Elimination, will be covered tomorrow. Tonight, on homework, we are going to solve by graphing. Your big job is to drive home the point that since a graph represents all the points where a particular relationship is true, therefore the place where the two graphs intersect is the point where both relationships are true. Also talk about the fact that graphing is not 100% accurate (you sort of eyeball a point and say "it looks like around this") and how to check your answer (plug it back into both equations).

Homework:

"Homework: Simultaneous Equations by Graphing"

3.2 Simultaneous Equations[2]

At the start of the second day, explain the two techniques of substitution and elimination. This is review, so they should get it with just a few examples.

Then have them do the assignment "Simultaneous Equations."

At the end of class, if they are mostly done, they can finish that for homework and also do the homework. If they are not mostly done, they can just finish it for homework, and you can have them do the "homework" the next day in class.

Homework:

"Homework: Simultaneous Equations"

[1]This content is available online at <http://cnx.org/content/m19497/1.2/>.
[2]This content is available online at <http://cnx.org/content/m19498/1.2/>.

3.3 The "Generic" Simultaneous Equation[3]

This should be done in class as another TAPPS exercise. Remind them of the ground rules, and especially of the fact that you will be asking them questions afterwards to make sure they got it.

3.3.1 Time for another test!

Once again, there is the sample test. If everyone finishes the TAPPS exercise early, and they all seem pretty comfortable with the material, you may not want to do the "do the sample test tonight and we'll go over it tomorrow and then have the test the next day"—tomorrow may be pretty boring! Instead, it may be OK (depending on the class) to just say "Now work in class on the sample test, use it to help you study tonight, and we will have a real test tomorrow."

[3]This content is available online at <http://cnx.org/content/m19499/1.2/>.

Chapter 4

Quadratics

4.1 Introduction[1]

There are really three separate pieces of this unit: factoring, solving quadratic equations, and graphing quadratic functions. The first piece is vital and important, but small. Nonetheless, you may want to add a small quiz between that section and the next. I have included two sample tests—the first on factoring and solving quadratic equations, the second on graphing.

4.2 Multiplying Binomials[2]

Sounds trivial, doesn't it? But this is one of the most important days in the year.

What they **do** know, from Algebra I, is how to FOIL. This takes two seconds of review and you're done. However, there are two points that their Algebra I teacher never made.

1. When we say $(x+3)(x+4) = x^2 + 7x + 12$, we are asserting the equality of two functions—that is, if I plug any number into $(x+3)(x+4)$, and plug that same number into $x^2 + 7x + 12$, it should come out the same. It's an **algebraic generalization**.
2. FOIL leaves you high and dry if you have to multiply $(x+2)(x+y+3)$. The **real** algorithm for multiplying polynomials is to multiply **everything on the left** by **everything on the right**. Walk through an example of this on the board. Show them how FOIL is just a special case of this rule, with both things are binomials.

At this point, they can start working in pairs on the exercise "Multiplying Binomials." They should have no problem with the first few. As you are walking around, your main job is to make sure that they are doing #5 correctly. They should **not** be multiplying these out explicitly (so that $(x+4)(x+4)$ becomes $x^2+4x+4x+16$ and then combining the middle terms. They should instead be using the **formula** that they just developed, $(x+a)^2 = x^2 + 2ax + a^2$ to jump straight to the right answer. A lot of them will find this very confusing. I always explain it this way: x and a are both placeholders that could represent **anything**. So when we say:

$(x+a)^2 = x^2 + 2ax + a^2$

what we're really saying is:

$(\text{something} + \text{something\_else})^2 = \text{something}^2 + 2(\text{something})(\text{something\_else}) + \text{something\_else}^2$

Maybe walk them through the first one as an example.

One point of this exercise is to get them to the point where they can see immediately, **with no in-between steps**, that $(x+4)^2 = x^2 + 8x + 16$. Some of them will think that this new, confusing method

[1] This content is available online at <http://cnx.org/content/m19469/1.2/>.
[2] This content is available online at <http://cnx.org/content/m19472/1.2/>.

may be faster, but they can just go right on doing it the "old way" with FOIL. I always explain to them that, in a few days, we'll be learning a technique called **completing the square** that involves **reversing** this formula, and therefore cannot possibly be done with FOIL. They need to know the formula.

Another point is to get our three formulae on the table: $(x+a)^2$, $(x-a)^2$, and x^2-a^2. There are very few things I ask the class to memorize during the year, but these three formulae should all be committed to memory.

But the larger point is to give them a new understanding and appreciation for what variables do—to understand that x and a represent **anything**, so that once you have a formula for $(x+a)^2$ you can use that formula directly to find $(2y+6z)^2$.

Homework:

"Homework: Multiplying Binomials"

4.3 Factoring[3]

Begin class by reminding them of what they already know: factoring means turning $x^2+7x+12$ into $(x+3)(x+4)$. Then ask—how would we **check** that? There are two ways. First, we can multiply it back (using FOIL for instance). Second, we can try a number (since we are making the claim that these two functions are "equal," remember?). Stress this very heavily: they have to know **both ways** of checking. Why? Because if you don't know both ways of checking, then you don't really understand what factoring is, even if you get the right answer.

OK, on to…how do you do it? There are three steps to factoring.

1. Pull out common terms. **This is always the first step!**
2. Use the formulae from yesterday. For instance, given x^2-9, you recognize it as the difference between two squares. Given x^2-6x+9, you recognize it as $(x-3)^2$.
3. When all else fails, plain old-fashioned factoring. Do a few examples on the board, just to refresh their memory. $x^2+7x+12$, $x^2-7x+12$, x^2+x-12, x^2-x-12 are good starting examples, and give you an opportunity to talk about what effect negative numbers have, both in the middle term (not much) and in the last term (lots). The thing I always stress is to **start with the last term**—find all the pairs of numbers that multiply to give you the last term, and then see if any of them add to give you the middle term. The real test is when you are faced with something like x^2+4x+8; it should **not take long** to determine that it cannot be factored!

Now they are ready to start on the "Factoring" assignment during class.

Homework:

"Homewoik: Factah Alla Dese Heah Spressions"

As I mentioned, this might be a good place to break and have a quiz. Or it might not. What do I know? Anyway, on to quadratics…

4.4 Introduction to Quadratic Equations[4]

Get them started on the assignment "Introduction to Quadratic Equations" with little or no preamble. Then, after a few minutes—after everyone has gotten through #5—stop them.

Make sure they all got the right answers to numbers 2 and 3, and that they understand them. If $xy=0$ then **either** x, **or** y, **must equal zero**. There is no other way for it to happen. On the other hand, if $xy=1$, that doesn't tell you much—either one of them could be anything (except zero).

Now, show how this relates to quadratic equations. They do remember how to solve quadratic equations by factoring. $x^2+x--12=0$, $(x+4)(x-3)=0$, $x=-4$ **or** $x=3$. But that last step is taken as a random

[3] This content is available online at <http://cnx.org/content/m19466/1.2/>.

[4] This content is available online at <http://cnx.org/content/m19470/1.2/>.

leap, "because they told me so." The thing I want them to realize is that, when they write $(x+4)(x-3)=0$, they are in fact asserting that "these two numbers multiply to give zero," so one of them has to be zero. This helps reinforce the idea of the previous lesson, that x and y can mean **anything**: $(x+4)(x-3)=0$ is in fact a special case of $xy=0$.

The acid test is, what do you do with $(x+5)(x+3)=3$? The ones who don't get it will turn it into $x+5=3$, $x+3=3$. And get two wrong answers. Instead you have to multiply it out, **then** get everything on one side so the other side is 0, and **then** factor.

Now they can keep going. Many of them will need help with #6—talk them through it if they need help, but make as much of it as possible come from them. This is a very standard sort of "why we need quadratic equations" type of problem.

The last four problems are a sneaky glimpse ahead at completing the square. For #11, many students will say $x=3$; remind them that it can also be –3. For #12, this is yet another good example of the "x can be anything" rule, and should remind them in some ways of the work we did with absolute values: something$^2=9$, so something $=\pm3$. #13 is obviously #12 rewritten, and #14 can be turned into #13 by adding 16 to both sides.

Homework:

"Homework: Introduction to Quadratic Equations."

4.5 Completing the Square[5]

When you're going over the homework, talk for a while about the throwing-a-ball-into-the-air scenario. It will come up again, and I really want people to understand it. The particular point I try to make is how the math reflects the reality. You have a function $h(t)$ where if you plug in any t at all, you will get an h. You're using it backward, specifying h and asking for t (as in, "when will the ball hit the ground?"). What kind of answers would you expect? Well, suppose you throw the ball 16 ft in the air. If you ask "When will it be at 20ft?" you would expect to get no answer at all. If you ask "When will it be at 5 ft?" you would expect two answers—one on the way up, and one on the way down. If you ask "When will it be at 16 ft?" you would expect exactly one answer. In all three cases, the math gives you exactly what you expect.

On the other hand, suppose you ask "When will it be at –3 ft?" (That is, under the ground.) You might expect no answer at all, since the ball never is under the ground. But the math doesn't know that—it thinks the ball is following the same function forever. So you get two answers. One is **after** the ball hits the ground. The other is before it left—a negative time! This is where you have to use common sense to find the "real" answer, as distinct from the answer the math gave you.

I spend a good half-period, at least, talking through this. I think it is an incredibly important point about the way we use math to model the world. See this webpage[6] for an exercise you can use just on this.

Anyway, onward…the assignment "Completing the Square" pretty much speaks for itself. Probably the only preamble you need is to point out that many quadratic equations, which **do have solutions**, cannot be factored. So we are going to learn another technique which has the advantage that it can **always** be used. (Factoring is still easier and faster when it works.)

Now you can just get them started on it, and then wander around and help. Just make sure that before the class is done, everyone gets the technique. You may also want to point out to them that they already did this on yesterday's assignment.

On #4 make sure they get two answers, not just one!

Homework:

"Homework: Completing the Square". The hard ones here, that you will get questions on the next day, are #9 and #10. Note that, on #9, I am **not** looking for the discriminant and the quadratic formula and stuff; just the obvious fact, based on completing the square, that if $c<0$ we have no real answers, if $c=0$ we have one, and if $c>0$ we have two. #10 is worth looking at closely if there are questions, because it leads to the next day.

[5]This content is available online at <http://cnx.org/content/m19465/1.2/>.
[6]http://www.ncsu.edu/felder-public/kenny/papers/physicist.html

4.6 The "Generic" Quadratic Equation[7]

Begin by reminding them of what we did with simultaneous equations. First, we learned how to solve them (using substitution or elimination). Then we used those exact same techniques to solve the **generic** version—that is, simultaneous equations where all the numbers were replaced by letters. This, in turn, gave us a **formula** that could instantly be used to solve any pair of simultaneous equations.

Now we are going to do that same thing with quadratic equations. The "generic" quadratic equation is, of course, $ax^2 + bx + c = 0$. Now, we have learned two different ways of solving such equations. The "generic" version is hard to solve by factoring (although it is possible); we are going to do it by completing the square.

Make sure they look over my example of completing the square; this might be a good opportunity for a quick TAPPS exercise. There are other examples in the "Conceptual Explanations" so you could do two TAPPS exercises—that way everyone gets a chance to be the teacher.

Then have them work through the sheet. They should derive the quadratic formula, and then use it.

By the time they are done, they should have two things. They should have the quadratic formula, and a bit of practice using it—so now we have **three** different techniques for solving quadratic equations. They should also have **derived** the formula. I always warn them that I will ask for this derivation on the next test: it is not enough to know the formula (although that too is good), you have to be able to derive it.

At the end of class, you may want to talk for just a couple of minutes about the discriminant, in reference to #11. It should be fairly obvious by that point to most of them.

Homework:

"Homework: Solving Quadratic Equations"

4.6.1 Time for another test!

As always, there is the sample test, which may or may not be assigned as a homework. Then there is the test—on multiplying polynomials, on factoring, and mostly on solving quadratic equations. Make it shorter than my sample [U+263A]

4.7 Graphing Quadratic Functions[8]

OK, we're done solving quadratic equations—we already have three techniques and that's enough. But—one thing I say a million times throughout my class—you never really understand a function until you graph it.

So, they can do the exercise "Graphing Quadratic Functions." It doesn't require any buildup, they can do it right now. Note that we are not introducing any of the formal machinery of parabolas (focus, directrix, *etc.)*—all that will come much later, in the unit on conics. We are graphing both horizontal and vertical parabolas the way we did in the very first unit on functions—by taking an initial starting point ($y = x^2$ or $x = y^2$) and moving it up and down and left and right and stretching it and turning it upside-down. None of this should require a calculator.

It might be worth mentioning that a horizontal parabola is not a function. But we can still talk about it and graph it.

Homework:

Finish the in-class assignment. That was a long one, wasn't it?

4.8 Graphing Quadratic Functions II[9]

The beginning of the "Graphing Quadratic Functions II" exercise is review of yesterday. After letting them work on it together, you may want to interrupt and have them do the thing in the middle as a TAPPS

[7]This content is available online at <http://cnx.org/content/m19480/1.2/>.
[8]This content is available online at <http://cnx.org/content/m19468/1.2/>.
[9]This content is available online at <http://cnx.org/content/m19467/1.2/>.

exercise. The key things you need to ask them about are how this is the same as, and different from, the way we completed the square before. For instance, we used to add nine to both sides (because, let's face it, we **had** two sides). Now we only have one side, so we add 9 to it, **and** subtract 9 from it, at the same time. This gives us what we want (the perfect square) without changing the function.

Homework:

"Homework: Graphing Quadratic Functions II." You will get questions the next day about #8 (which is really a line) and #11 (which they just flat can't graph at this point). These lead nicely into #12. If it has an x^2 but no y^2, it's a vertical parabola. If it has a y^2 but no x^2, it's a horizontal parabola.

4.9 Solving Problems by Graphing Quadratic Functions[10]

Now, at long last, we see a **use** for all this graphing we've been doing.

In the throwing-a-ball scenario, we have an $h(t)$ that can be used to answer two kinds of questions. "I know the time, but what is the height?" (easy, plug in) and "I know the height, what is the time?" (harder, requires solving a quadratic equation). But there is a third kind of question, very important in the real world, which is: "How high does it go?" Now we don't know the time **or** the height! But if we graph it, and find the vertex, we can find both.

Now they can work a while on the in-class assignment. Many of them will get stuck dead on #3. This is where you have to pull back and lecture a bit more. Help them draw it, and set up the function $A(x)$. But more importantly, talk about what that function **means**. You plug in any x (length) and you get back an A (area). So, if the graph looks like this $\cap$ what does that tell us? Well, at the peak there, that is the highest A ever gets on our graph—that is, the highest the area ever gets. Find the vertex, and you will find the x that maximizes A!

This is worth a lot of time to make sure people really get it. It comes all the way back to week 1, and the idea of graphing a function. On one level, it's incredibly abstract—we are drawing an upside-down parabola that somehow represents the "possibility spaces" for a bunch of rectangles. But if you understand the idea of graphing a function, it is really very simple. Every point on that parabola pairs an x (length) with an A (area). Every point represents one farm that our farmer could create. It's obvious, looking at it, that this point at the top here represents the one with the **highest area**.

This is one of those cases where the in-class assignment and the homework, together, could easily take two days instead of one. Let it take that, if it does. Make up more problems, if you have to. But don't let them get away with thinking "I understand everything else, I just don't get the word problems."

Homework:

"Homework: Solving Problems by Graphing Quadratic Functions"

4.10 Quadratic Inequalities[11]

For some reason, this is one of the hardest topics in the course. It shouldn't be hard. There is nothing hard about it. But students get incredibly tied in knots on this, by trying to take short cuts. The hardest part is convincing them that they have to think about it graphically.

So, begin by simply putting these two problems on the board.

$x^2 - -3x - -4 > 0$

$x^2 - -3x + 3 > 0$

Allow them to work in pairs or groups. Offer a piece of candy or a bit of extra credit or some such to anyone who can find the answer to both problems. Give them time to really work it. Almost no one will get it right, and that's the point. It's very hard to think about a problem like this algebraically. It's very easy if you think about it the right way: by graphing.

[10] This content is available online at <http://cnx.org/content/m19479/1.2/>.
[11] This content is available online at <http://cnx.org/content/m19473/1.2/>.

So, we're going to graph both of those functions. But strangely enough, we're going to do it without completing the square or finding the vertex. In each case, we're only going to ask two questions: what are the **zeros** of the function, and which **direction** does it open in? These two questions are all we need to answer the inequality.

In the first case, by factoring, we find two zeros: 4, and –1. In the second case, we find with the quadratic formula that there are no zeros. Both graphs open up. (Why? Because **the coefficient of the** x^2 **term is positive**.) So the graphs look something like this.

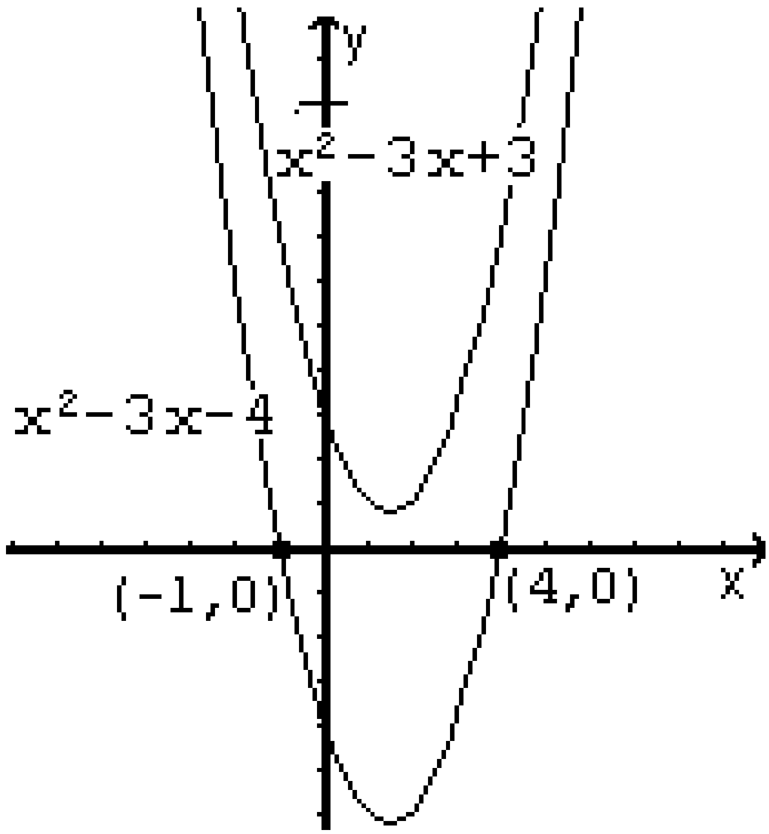

Figure 4.1

What are the vertices, exactly? We don't know. If we wanted to know that, we would have to complete the square, just as we did before.

But if all we want to know is **where each graph is positive**, we now have it. The graph of the first function should make it clear that all numbers to the **right of** 4 work, as do the numbers to the **left of** –1, but the numbers in between don't work. (Quick review: how can we write that answer with inequalities? With set notation?) The graph of the second function makes it clear that **all numbers work**.

Check this by trying numbers in the original inequalities.

Now, challenge them to find a quadratic inequality that is in the form $f(x) > 0$ where $f(x)$ is a quadratic function and the solution is **nothing works**. The key is, of course, it has to be an upside-down quadratic.

Homework:
"Homework: Quadratic Inequalities"

4.10.1 Time for another test II!

Congratulations, kids! We are done with our entire unit on quadratic equations—probably the biggest unit in the course. (Certainly the only think I remember from my own Algebra II course.)

Chapter 5

Exponents

5.1 Introduction[1]

After all the incredibly **new** stuff we've been doing, it's a nice break to get back to something with a large element of review in it.

But it is also a problem. Many of the kids know already that $2^3 2^5 = 2^8$. A fair number of them even know that $2^{-7} = 1/27$. But they don't know **why**. It's vital to keep reminding them that it isn't enough to know it, they have to know why—and this will indeed be reflected on the test.

5.2 Rules of Exponents[2]

Yes, a lot of this assignment was already done, verbatim, in the unit on Functions. But there are a lot of reasons for bringing it back (just as we did in Quadratics). First, they (and you) can discover that they have gotten better at finding generalizations. But more importantly, back when we did this in functions, we were only interested in the process of finding generalizations. Now we are focused on creating, memorizing, and using the three rules of exponents.

2d is really pushing them another step toward "the way mathematicians think"—seeing that $2\left(2^x\right) = 2^{x+1}$ is really just a special case of $2^a 2^b = 2^{a+b}$, where $a = 1$.

In #3, I really want to get at the idea that $\frac{3^{12}}{3^{10}} = 3^2$, and $\frac{3^{10}}{3^{12}} = \frac{1}{3^2}$. In other words, the whole thing can and should be done without negative exponents. Why? Because we haven't yet defined what they mean—and why.

Homework:

"Homework: Rules of Exponents"

5.3 Extending the Idea of Exponents[3]

This is one of my favorite class discussions. You're going to talk almost the entire class, and just give out an assignment toward the end.

So far, we have only talked about exponents in the context of positive integers. The base can be anything: for instance, we can find $(-3)^4$ or $\left(\frac{1}{2}\right)^3$. But when we say that 2^3 means $2 \bullet 2 \bullet 2$, that definition is really only meaningful if the exponent is a positive integer. We can't multiply 2 by itself " $--3$ times" or "$\frac{1}{2}$ times." Or "0 times" for that matter.

[1]This content is available online at <http://cnx.org/content/m19325/1.2/>.
[2]This content is available online at <http://cnx.org/content/m19327/1.3/>.
[3]This content is available online at <http://cnx.org/content/m19318/1.2/>.

So, let's plop ourselves down in an imaginary point in history where exponents are only defined for positive integers. We are the king's mathematicians. The king has just walked in and demanded that we come up with some sort of definition for what 2^{-3} means. "Zero and negative numbers have rights too," he growls. "They must be treated equally, and given equal rights to be in the exponent."

So we start with a brainstorming exercise. The object is to come up with as many possible things as you can think of, for 2^{-3} to mean. As always, this should be done in groups of 2 or 3, and remind them that the object is quantity, not quality: let's get creative. If you can't think of more than two or three definitions, you're not trying hard enough.

By the way, somewhere in class there is a smart-aleck who knows the right answer and therefore won't plan. "It's $1/2^3$" he insists proudly. "Why are we doing this?" To which you reply: "The people in the class are coming up with dozens of things it could mean. Can you give them a good argument as to why it should mean that, instead of all the others?" And he weakly answers "Well, my Algebra I teacher told me..." and you've won. The point, you explain, is not to parrot what your teacher told you, but to understand several things. The first is that our old definition of an exponent ("multiply by itself this many times") just doesn't apply here, and so we need quite literally a new definition. The second is that there are a ton of definitions we could choose, and it frankly seems arbitrary which one we pick. The third is that there really is a good reason for choosing one and only one definition. But if he doesn't know what that he, how about if he gets with the program and brainstorms? End of discussion.

OK, so after a few minutes, you start collecting ideas from the groups. 2^{-3} means... 2^3, only negative (so it's –8). It means 2 divided by itself three times, 2/2/2 (so it's either 2 or $\frac{1}{2}$, depending on how you parenthesize it). And so on. With a bunch of ideas on the board, you say, now we have to choose one. How do we do that? That is, what criteria do we use to decide that one definition is better than the others? (silence)

The answer is—the definition should be as consistent as possible with the one we already have. Of course it won't mean the same thing. But it should behave mathematically consistently with the rules we have: for instance, it should still obey our three laws of exponents. That sort of consistency is going to be the guideline that we use to choose a definition for the king.

And hey, what exactly do negative numbers mean anyway? One way to look at them is, they are what happens when you take the positive numbers and keep going down. That is, if you go from 5 to 4 to 3 and just keep going, you eventually get to 0 and then negative numbers. This alone is a very powerful way of looking at negative numbers. You can use this to see, for instance, why positive-times-negative-equals-negative and why negative-times-negative-equals-positive.

$3 \bullet 5 = 15$		$3 \bullet -5 = -15$
$2 \bullet 5 = 10$		$2 \bullet -5 = -10$
$1 \bullet 5 = 5$		$1 \bullet -5 = -5$
$0 \bullet 5 = 0$		$0 \bullet -5 = 0$
$-1 \bullet 5 = -5$		$-1 \bullet -5 = 5$
$-2 \bullet 5 = -10$		$-2 \bullet -5 = 10$

Table 5.1

OK, you may or may not want to get into this, but I think it's cool, and it does help pave the way for where we're going. (It also helps reinforce the idea that even the rules you learned in second grade have reasons.) The numbers I've written on the left there show what happens to "multiply-by-5" as you count down. Clearly, looking at the positive numbers, the answers are going down by 5 every time. So if that trend continues as we dip into negative numbers, then we will get –5 and then –10 on the bottom: negative times positive equals negative.

On the right, we see what happens to "multiply-by–5" as you count down. Since we already know (just proved) that positive-times-negative-equals-negative, we know that $3 \bullet -5 = -15$ and so on. But what is happening to these answers as we count down? They are going up by 5. So, continuing the trend, we find that $-1 \bullet -5 = 5$ and so on.

As I say, you may want to skip that. What is essential is to get across the point that we need a new definition that will cover negative exponents, and that we are going to get there by looking for consistency with the positive ones. Then they are ready for the in-class assignment: it shouldn't take long. Do make sure to give them 10 minutes for it, though—you want them to finish it in class, and have time to ask you questions, so you know they are ready for the homework.

Homework:

"Homework: Extending the Idea of Exponents"

5.4 Fractional Exponents[4]

Start by reminding them of where we are, in the big picture. We started with nothing but the idea that exponents mean "multiply by itself a bunch of times"—in other words, 7^4 means $7 \bullet 7 \bullet 7 \bullet 7$. We went from there to the rules of exponents— $x^a x^b = x^{a+b}$ and so on—by common sense. Then we said, OK, our definition only works if the exponent is a positive integer. So we found new definitions for zero and negative exponents, but extending down from the positive ones.

Now, we don't have a definition for fractional exponents. Just as with negative numbers, there are lots of definitions we could make up, but we want to choose one carefully. And we can't get there using the same trick we used before (you can't just count and "keep going" and end up at the fractions). But we still have our rules of exponents. So we're going to see what sort of definition of fractional exponents allows us to keep our rules of exponents.

From there, you just let them start working. I can summarize everything on the assignment in two lines.

1. The rules of exponents say that $\left(x^{\frac{1}{2}}\right)^2 = x$. So whatever $x^{\frac{1}{2}}$ is, we know that when we square it, we get x. Which means, by definition, that it must be $\sqrt{x}$. Similarly, $x^{\frac{1}{3}} = \sqrt[3]{x}$ and so on.
2. The rules of exponents say that $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$. Since we now know that $x^{\frac{1}{3}} = \sqrt[3]{x}$, that means that $x^{\frac{2}{3}} = \left(\sqrt[3]{x}\right)^2$. So there you have it.

Thirty seconds, written that way. A whole class period to try to get the students to arrive their on their own, and even there, many of them will require a lot of help to see the point. Toward the end, you may just call the class's attention to the board and write out the answers. But by the time they leave, you want them to have the following rule: for fractional exponents, the denominator is a root and the numerator is an exponent. And they should have some sense, at least, that this rule followed from the rules of exponents.

There is one more thing I really want them to begin to get. If a problem ends up with $\sqrt{25}$, you shouldn't leave it like that. You should call it 5. But if a problem ends up with $\sqrt{2}$, you should leave it like that: don't type it into the calculator and round off. This is also worth explicitly mentioning toward the end.

Homework:

"Homework: Fractional Exponents". Mostly this is practicing what they learned in class. The inverse functions are a good exercise: it forces them to review an old topic, but also forces them to practice the current topic. For instance, to find the inverse function of y=x2/3, you write:

$x = y^{\frac{2}{3}} = \sqrt[3]{y^2}$. $x^3 = y^2$. $y = \sqrt{x^3} = x^{\frac{3}{2}}$.

After you go through that exercise a few times, you start to see the pattern that the inverse function actually inverts the exponent. The extra fun comes when you realize that x^0 has no inverse function, just as this rule would predict.

At the end of the homework, they do some graphs—just by plotting points—you will want to make sure they got the shapes right, because this paves the way for the next topic. When going over the homework the next day, sketch the shapes quickly and point out that, on the graph of 2^x, every time you move on to the right, y doubles. On the graph of $\left(\frac{1}{2}\right)^x$, every time you move one to the right, y drops in half.

[4]This content is available online at <http://cnx.org/content/m19322/1.2/>.

5.5 "Real Life" Exponential Curves[5]

This is another one of those topics where the in-class exercise and the homework may take a total of two days, combined, instead of just one. This is a difficult and important topic.

We begin with a lecture something like the following:

Earlier this year, we talked about "linear functions": they add a certain amount every time. For instance, if you gain $5 every hour, then the graph of your money vs. time will be a line: every hour, the total will add 5. The amount you gain each hour (5 in this case) is the slope.

Can a line also subtract every day? Sure! That isn't a different rule, because adding is the same as subtracting a negative number. So if Mr. Felder is losing ten hairs a day, and you graph his hairs vs. time, the graph will be a line going down. The total subtracts 10 every day, but another way of saying that is, it adds –10 every day. The slope is –10. This is still a linear function.

So why am I telling you all this? Because "exponential functions" are very similar, except that they multiply by the same thing every time. And, just as linear functions can subtract (by adding negative numbers), exponential functions can divide (by multiplying by fractions: for instance, multiplying by $\frac{1}{3}$ is the same as dividing by 3). The amount you multiply by is called…well, come to think of it, it doesn't have a cool name like "slope." I guess we could call it the "base."

Then they can begin to work on the assignment. They will make it through the table all right. But when it comes to finding the formula for the nth day, many will fall down. Here is a way to help them. Go back to the table and say: "On day 3, let's not write "4"—even though it is 4 pennies. It is 2 times the previous amount, so let's just write that: 2×2. On day 4, it's 2 times that amount, or $2 \times 2 \times 2$. On day 5, it's 2 times that amount, or $2 \times 2 \times 2 \times 2$. This is getting tedious…what's a shorter way we can write that?" Once they have expressed every answer in powers of 2, they should be able to see the 2^{n-1} generalization. If they get the wrong generalization, step them through to the next paragraph, where they test to see if they got the right answer for day 30.

You go through the same thing on the compound interest, only harder. A lot of hand-holding. If you end one year with x then the bank gives you $.06x$ so you now have a total of $x + .06x$ which is, in fact, $1.06x$. So, hey, your money is multiplying by 1.06 every year! Which means if you started with $1000 then the next year you had $\$1000 \bullet 1.06$. And the next year, you multiplied that by 1.06, so then you had $\$1000 \times 1.06 \times 1.06$. And the year after that….

Toward the end of class, put that formula, $\$1000 \times 1.06n$, on the board. Explain to them that they can read it this way: "Just looking at it, we can see that it is saying you have $1000 multiplied by 1.06, n times." This is always the way to think about exponential functions—you are multiplying by something a bunch of times.

The assignment is also meant to bring out one other point that you want to mention explicitly at the end. When we developed our definitions of negative and fractional exponents, we wanted them to follow the rules of exponents and so on. But now they are coming up in a much more practical context, and we have a new need. We want $x^{2\frac{1}{2}}$ to be bigger than x^2 and smaller than x^3, right? After all, after $2\frac{1}{2}$ years, you certainly expect to have more money than you had at the beginning of the year! It isn't obvious at all that our definition, $x^{\frac{5}{2}} = \sqrt{x^5}$, will have that property: and if it doesn't, it's useless in the real world, even if it makes mathematicians happy. Fortunately, it does work out exactly that way.

Homework:

"Homework: 'Real life' exponential curves"

5.5.1 Time for Another Test!

The sample test will serve as a good reminder of all the topics we've covered here. It will also alert them that knowing why $x^{\frac{1}{2}}$ is defined the way it is **really does count**. And it will give them a bit more practice (much-needed) with compound interest.

[5]This content is available online at <http://cnx.org/content/m19329/1.3/>.

Chapter 6

Logarithms

6.1 Introduction[1]

I talk to a surprising number of math teachers who are really uncomfortable with logs. There's something about this topic that just makes people squeamish in Algebra, in the same way that "proving a series converges" makes people squeamish in Calculus.

It doesn't have to be hard. It is not intrinsically more complicated than a radical. When you see $\sqrt[3]{x}$ you are seeing a mathematical question: "What number, raised to the 3rd power, gives me x?" When you see $\log_3 x$ you are seeing a question which is quite similar: "3, raised to what power, gives me x?" I say this about a hundred times a day during this section. My students may forget the rules of logs and they may forget what a common log is and they will almost certainly forget e, but none of them will forget that $\log_2 8$ means the question "2 to what power is 8?" You may want to show them the "Few Quick Examples" at the beginning of the Conceptual Explanations chapter to drive the point home.

It is possible to take any arbitrary logarithm on a standard scientific or graphing calculator. I deliberately never mention this fact to my students, until the entire unit (including the test) is over. Faced with $\log_2 8$ I want them to think it through and realize that the answer is 3 because $2^3 = 8$. The good news is, none of them will figure out how to do that problem on the calculator, if you don't tell them.

6.1.1 Introduction to Logarithms

This is a pretty short, self-explanatory exercise. There isn't anything you need to say before it. But you do need to do some talking after the assignment. Introduce the word "log" and explain it, as I explained it above: $\log_2 8$ means "2 to what power is 8?" Also discuss the fact that the log is always the inverse of the exponential function.

After they have done the assignment, and heard your explanation of the word log, then they are ready for the homework. It wouldn't hurt if that happens in the middle of the class, so they can get started on the homework in class, and finish it up at home. The in-class exercise is short, the homework is long.

"Homework: Logs"

When going over the homework the next day, #20 can be explained two ways. First: 5 to what power is 5^4? When asked that way, it's easy, isn't it? You don't have to find what 5^4 is, to see that the answer is 4! But there is also another way to explain it, which gets back to the idea of 5^x and $\log_5 x$ being inverse functions. The first function turns 4 into 5^4. So the second one has to reverse this process, and turn 5^4 back into 4. This way is harder to understand, but it makes it a lot easier to see why #21 also has to be 4.

Then, there is the graph—as always, make sure they get the right general shape. Point out that the most salient feature of this graph is that it grows…incredibly…slowly as you go farther out to the right. (Every time x doubles, the graph just goes up by 1.) This is a lot of what makes logs useful, as we will see.

[1] This content is available online at <http://cnx.org/content/m19436/1.2/>.

6.2 Properties of Logarithms[2]

This is very standard stuff. Using the in-class exercise in groups of 2 or 3, they should be able to find—in some cases with a bit of help from you—some rules of logarithms. In the homework, they practice using those rules.

One thing I don't do in the worksheets is formally prove the rules. However, I have been known to "throw in" the proofs sometimes in class, either for a group that finishes early, or for the whole class if enough people are interested. One of the proofs is provided as an example in the "Conceptual Explanations" along with guidelines for the other two.

But what I really care about is giving them an intuitive grasp of why the rules work, rather than the proof. The intuitive grasp is what comes from the exercise, from realizing that the logarithm is essentially a counter. Once you see that $\log_2 8$ is asking how many 2s there are in 8, then it's obvious that $\log_2 (8 \bullet 16)$ will add up all the 2s in 8, and in 16.

Homework:
"Homework: Properties of Logarithms"

6.3 Using the Laws of Logarithms[3]

Once you have gone through the laws of logarithms, you can spend five minutes working a couple of problems on the board, like:

$\log_5 (x) = \log_5 (3)$

and then

$\log_5 (x+1) + \log_5 (x-1) = \log_5 (8)$

The first establishes that if you have $\log_{(\text{this})} = \log_{(\text{that})}$, then this must equal that. The second shows how you have to use the laws of logs to get into that form. (The OK students will answer 3. The better students will answer ± 3. Only the very best will get ± 3 and then realize that the -3 is, after all, invalid! But all of that is a detail, of course.)

Anyway, then there is the worksheet full of problems like that, which also gives good review of a number of old topics.

The thing is, this really isn't a whole day. Sneak it in when you have 15-20 minutes left in class. It doesn't matter whether it comes before, after, or in the middle of the next topic.

6.4 So What Are Logarithms Good For, Anyway?[4]

As always, things get harder when we get into word problems. There are a few things I want them to take away here.

First—logs are used in a wide variety of real world situations.

Second—logs are used because they compress scales. In other words, because they grow so slowly, we use logarithmic scales whenever we want to work with a function that, by itself, grows too quickly. Or, to put it another way, we use logarithms whenever something varies so much that you don't care exactly what is, just what the power of 10 is. Don't say all this before they start working, but hopefully they will come up with something like this on #6.

Homework:
"Homework: What Are Logarithms Good For, Anyway?"

In addition to following up on the in-class work, the homework here also introduces the common and natural logs. It's a bit of a weak connection, but I had to stick them somewhere.

[2]This content is available online at <http://cnx.org/content/m19438/1.1/>.
[3]This content is available online at <http://cnx.org/content/m19440/1.1/>.
[4]This content is available online at <http://cnx.org/content/m19439/1.2/>.

6.4.1 Time for Another Test!

The sample test is actually pretty important here. It pulls together a lot of ideas that have been covered pretty quickly.

The extra credit is just a pun. The answer is log cabin or, better yet, natural log cabin. Who says math can't be fun?

According to my reckoning, you are now approximately halfway through the curriculum. Mid-terms are approaching. If there are a couple of weeks before mid-terms, I would not recommend going on to radicals—spend a couple of weeks reviewing. Each topic (each test, really) can stand a whole day of review. It may be the most important time in the whole class!

Chapter 7

Rational Expressions

7.1 Introduction[1]

We've talked about the word "rational"—it doesn't mean "sane," it means a "ratio" or, in other words, a fraction. A rational expression is just a fraction with variables.

This section is unique, perhaps, in the fact that it introduces practically no new skills. They have to be able to factor; they have to know the rules of exponents; they have to be able to work with fractions; they even have to be able to do long division. There is nothing new in any of that. It's just putting it all together to simplify, and work with, rational expressions.

Part of the benefit of this unit is that there are always a few kids in class—maybe more than a few—who have a lingering, secret fraction-phobia. They are hoping that no one will ever notice because the calculator will always rescue them. You can spot these people because they always answer everything—including "what is 2 divided by 3?"—in decimals. But this unit will flush them out. You can't get through rational expressions unless you know how to do fractions, and your calculator will not help you. (I always point this out, very explicitly, several times.) In the "Conceptual Explanations" I begin each section by working plain-old-number-fraction problems (simplifying them, multiplying them, adding them, and so on); tell them they can look there if they want a quick review.

Because of the nature of this unit—no new concepts, and fraction phobia—it has fewer "creative thinking" types of problems, and more "drill and practice," than any other unit. It gets boring for you, but don't let them see that. For a few students at least, this has the potential to break down a barrier that they have been struggling with since the third grade.

7.2 Rational Expressions[2]

Begin by explaining what rational expressions are, and making the points I made above—we are going to put together some of our old skills in a new way, which will require us to be good at fractions. So, we're going to start by reviewing how to work with fractions.

Then pair the students up. We're going to do a sort of do-it-yourself TAPPS exercise. One partner is the student, one is the teacher. The teacher's job is to add $\frac{1}{2}+\frac{1}{3}$—not just to come up with the answer, but to walk through the process, explaining what he is doing and why he is doing it at every step, all on paper. The student asks for clarifications of any unclear points. By the time they are done, they should have a written, step-by-step instruction guide for adding fractions.

Then they switch roles. The former student becomes the new teacher, and gives two lessons: how to multiply $\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)$ and how to divide $\frac{1}{2}/\frac{1}{3}$. Once again, they should wind up with a step-by-step guide.

[1]This content is available online at <http://cnx.org/content/m19486/1.1/>.
[2]This content is available online at <http://cnx.org/content/m19488/1.1/>.

The original teacher takes over again, and shows how to simplify a fraction.

Finally, you give a brief lesson on multiplying fractions—they've already done that, but the key point to emphasize here is that you can cancel before you multiply. For instance, if you want to multiply $\frac{7}{8}$ times $\frac{10}{21}$, you could say:

$\frac{7}{8} \times \frac{10}{21} = \frac{70}{168}$

and then try to simplify that. But it's a lot easier to simplify before you multiply. The $\frac{7}{21}$ becomes $\frac{1}{3}$, the $\frac{10}{8}$ becomes $\frac{5}{4}$, so we have:

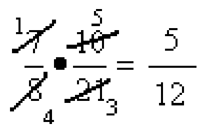

Figure 7.1

Of course, you can only do this trick—canceling across different fractions—when you are multiplying. Never when you are adding, subtracting, or dividing!

So why are we going through all this? Because, even though they know how to do it with numbers, they are going to get confused when it comes to doing the exact same thing with variables. So whenever they ask a question ("What do I do next?" or "Do I need a common denominator here?" or some such), you refer them back to their own notes on how to handle fractions. I have had a lot of students come into the test and immediately write on the top of it:

$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

They did this so they would have a "template" to follow when adding rational expressions—it's a very smart move.

Other than basic fraction manipulation, there is only one other big thing to know about rational expressions—always factor first. Factoring shows you what you can cancel (especially when multiplying), and how to find the least common denominator (when adding or subtracting).

So, walk through some sample problems for them on the blackboard. Put up the problem, and ask them what the first step is. . .and what the second step is. . .and so on, until you have something like this on the blackboard.

$\frac{1}{x} + \frac{2}{y} = \frac{y}{\text{xy}} + \frac{2x}{\text{xy}} = \frac{y+2x}{\text{xy}}$

Emphasize over and over that this is just the same steps you would take to add $\frac{1}{2} + \frac{2}{3}$. But at the end, point out one other thing—just as we did in the very first week of class, we have asserted that two functions are equal. That means they should come out exactly the same for any x and y. So have everyone choose an x-value and a y-value, and plug them into both $\frac{1}{x} + \frac{2}{y}$ and $\frac{y+2x}{\text{xy}}$, and make sure they come out with the same number.

Now walk through something harder on the blackboard, like this:

$\frac{x}{x^2+5x+6} - \frac{2x}{x^3-9x}$	The original problem
	continued on next page

$\frac{x}{(x+3)(x+2)} - \frac{2x}{x(x+3)(x-3)}$	Always factor first!
$\frac{x}{(x+3)(x+2)} - \frac{2}{(x+3)(x-3)}$	Simplify (cancel the "x" terms on the right).
$\frac{x(x-3)}{(x+3)(x+2)(x-3)} - \frac{2(x+2)}{(x+3)(x-3)(x+2)}$	Get a common denominator. This step requires a lot of talking through. You have to explain where the common denominator came from, and how you can always find a common denominator once you have factored.
$\frac{\left(x^2-3x\right)-(2x+4)}{(x+2)(x+3)(x-3)}$	Now that we have a common denominator, we can combine. This step is a very common place to make errors—by forgetting to parenthesize the $(2x+4)$ on the right, students wind up adding the 4 instead of subtracting it.
$\frac{x^2-5x-4}{(x+2)(x+3)(x-3)}$	Done! Of course, we could multiply the bottom through, and many students want to. I don't mind, but I don't recommend it—there are advantages to leaving it factored.

Table 7.1

Once again, have them try numbers (on their calculators) to confirm that $\frac{x}{x^2+5x+6} - \frac{2x}{x^3-9x}$ gives the same answer as $\frac{x^2-5x-4}{(x+2)(x+3)(x-3)}$ for any x. But they should also remember (from week 1) how to find the domain of a function—and in this case, the two are not quite the same. The function we ended up with excludes $x=-2$, $x=-3$, and $x=3$. The original function excludes all of these, but also $x=0$. So in that one case, the two are not identical. For all other cases, they should be.

Whew! OK, you've been lecturing all day. If there are 10 minutes left, they can begin the exercise. They should work individually (not in groups or pairs), but they can ask each other for help.

Homework:

Finish the in-class exercise and do "Homework—Rational Expressions"

7.3 Rational Equations[3]

After you have answered all the questions on the previous homework, they can just get started on this assignment immediately—it should explain itself pretty well.

However, after about 5 minutes—when everyone has gotten past the first two problems—pull them back and talk to the whole class for a moment, just to make sure they get the point. The point is that if the denominators are the same, then the numerators must be the same (#1); and if the denominators are not the same, then you make them the same (#2). It's pretty straightforward with these two problems, but it may be deceptively easy. The real thing to make sure they "get" is that, having established these two principals with these easy problems, they are now going to apply them in much more complicated ones.

Then they can get back to it, and you just float around and help. In #3, the only trick is remembering that if $x^2=25$, then $x=\pm 5$ (not just 5). Numbers 4 is straightforward. Give them time to struggle with #5 before pointing out that they should factor-and-simplify first—always factor first! #6 is really what all this is building up to: to solve rational equations in general, you must be able to solve quadratic equations!

Homework:

"Homework: Rational Expressions and Equations"

[3]This content is available online at <http://cnx.org/content/m19489/1.1/>.

7.4 Polynomial Division[4]

Half the class, maybe the whole class, will be lecture today—you have to show them how to do this. The lecture goes something like this.

Today we're going to talk about everybody's favorite topic. . .long division!

Before we do that, I have to start by pointing out some very important cases where you don't have to use long division. For instance, suppose you have this:

$$\frac{36x^3+8x^2+5x+10}{2x} \tag{7.1}$$

That problem doesn't require any hard work—you should be able to divide it on sight. (Have them do this.) You should have gotten:

$18x^2+4x+2\frac{1}{2}+\frac{5}{x}$

To take another example, how about this?

$$\frac{x^3-6x^2+5x}{x^2-5x} \tag{7.2}$$

That one isn't quite as easy. What do we do first? (factor!) Oh yes, let's do that! So it is $\frac{x(x-5)(x-1)}{x(x-5)}$. Oh, look. . .it's just $x-1$! Once again, no long division necessary.

OK, but suppose we had this?

$$\frac{6x^3-8x^2+4x-2}{2x-4} \tag{7.3}$$

How can we simplify it? This is where we're going to have to use. . .long division.

So, let's start the same way we started with fractions: by remembering how to do this with numbers. Everyone, at your seats, work out the following problem on paper.

$$\frac{4327}{11} \tag{7.4}$$

(Here you pause briefly while they work it on paper—then you work it on the blackboard.) OK, you should have gotten 393 with a remainder of 4. So the actual answer is 393 $\frac{4}{11}$. How could we check that? That's right. . .we would want to make sure that 393 $\frac{4}{11}$ times 11 gives us back 4327. Because that's what multiplication is—it's division, backward.

Now, let's go back to that original problem. OK, kids...I'm going to leave my number long division over here on the blackboard, and I'm going to work this rational expressions long division next to it on the blackboard, so you can see that all the steps are the same.

We'll start here:

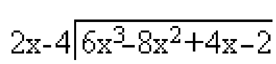

Figure 7.2

From there, develop the whole thing on the blackboard, step by step. (I do this exact problem in the "Conceptual Explanations." I would not suggest you tell them to look it up at this point; instead, I would recommend that you go through it on the blackboard, explaining steps as you go, and continually reinforcing

[4]This content is available online at <http://cnx.org/content/m19487/1.2/>.

the analogy to what you did with numbers.) In the end you conclude that $\frac{6x^3-8x^2+4x-2}{2x-4}$ is $3x^2+2x+6$ with a remainder of 22: or, to put it another way, $3x^2+2x+6+\frac{22}{2x+4}$.

So then you ask: OK, how could we test that? Hopefully they will come up with one answer, then you say "Good, how else?" and they come up with the other one. One way is to plug a number—any number—into $\frac{6x^3-8x^2+4x-2}{2x-4}$, and the same number into $3x^2+2x+6\frac{22}{2x-4}$, and make sure you get the same thing. The other way is to multiply back $3x^2+2x+6\frac{22}{2x-4}$ by $2x-4$ and make sure you get back to $6x^3-8x^2+4x-2$. Make sure they understand both ways. If you don't understand the first way, you don't understand function equality; if you don't understand the second way, you don't understand what division is!

Finally, you ask: how could we have made all that a bit easier? The answer, of course, is that we should have divided the top and bottom by 2 before we did anything else. This brings us back to our cardinal rule of rational expressions: always factor first!

If there is still time left in class, let them get started on the homework.

Homework:

"Dividing Polynomials"

When going over it, see how many people did #1 the "hard way." Remind them that if the bottom is only one term, you can just do the whole thing quickly and painlessly!

Optional Exercise:

If you have extra time—if some students get way ahead and you want to give them an extra assignment, or if you want to spend more time on this topic—here is a good exercise that brings things together. Suppose you want to solve the equation $6x^3-5x^2-41x-30=0$. You can use the "Solver" on the calculator and it will find one answer: most likely, $x=-1$. (This requires a 5-minute introduction to the "Solver.") How do you find the other answers?

Well, we recall from our study of quadratic equations that if $x=-1$ is an answer, then the original function must be expressible as $(x+1)$ (something). How do you find the something? With long division! (Maybe have them try it both ways.) What you end up with, after you divide, is $6x^2-11x-30$. You can factor that the "old-fashioned" way (which takes a bit of time) and you get $(2x+3)(3x-10)$ which gives you the other two roots.

7.4.1 Time for Another Test!

And we're done with yet another unit.

On #6 of the sample test, stress that they will get no credit without showing work. They can check their answer either way—multiplying back, or trying a number—but they have to show their work.

The extra credit is a good problem that I like to get in somewhere. I generally give one point for the obvious pairs (0,0) and (2,2), and two points for the equation $xy=x+y$. But what I really want to see is if they remember how to solve that for y and get $y=\frac{x}{x-1}$. Finally, from that, they should be able to see that $x=1$ has no pairing number (which is obvious if you think about it: nothing plus one gives you the same thing times one!).

Chapter 8

Radicals

8.1 Introduction[1]

Well, this is easier, isn't it? They know what a radical is. But they're going to go places with them that they have definitely never been before...

This looks long, but a lot of it is very fast. You don't have to do much setup, except to remind them what a square root is. I would explain it by analogy to the way we explained logs. $\log_2 8$ asks the question "2 to what power is 8?" Well, $\sqrt{9}$ also asks a question: "What squared is 9?"

But then, there is an important distinction—one that I like to make right away, and then repeat several times. The question "what squared is 9?" actually has two answers. So if we defined square root as the answer to that question, square root would not be a function—9 would go in, and both 3 and –3 would come out (the old "rule of consistency" from day 1). So we somewhat arbitrarily designate the $\sqrt{}$ symbol to mean the positive answer, so that it is a function. So if you see $x^2 = 9$ you should properly answer $x = \pm 3$. But if you see $x = \sqrt{9}$ then you should answer only $x = 3$. This is a subtle distinction, but I really want them to get it, and to see that it is nothing inherent in the math—just a definition of the square root, designed to make it single-valued. This is why if you see $x^2 = 2$ you have to answer $x = \pm\sqrt{2}$, to get both answers.

So, on to the assignment. It starts with a couple of word problems, just to set up the idea that radicals really are useful (which is not obvious). After everyone is done with that part, you may want to ask them to make up their own problems that require square roots as answers (they are not allowed to repeat #1). Get them to realize that we square things all the time, and that's why we need square roots all the time, whenever we want to get back. (We'll be returning to this theme a lot.)

Then there are problems with simplifying radicals. For many of them, they have seen this before—they know how to turn $\sqrt{8}$ into $2\sqrt{2}$. But they don't realize that they are allowed to do that because of the general rule that $\sqrt{ab} = \sqrt{a}\sqrt{b}$. So it's important for them to get that generalization, but it's also important for them to see that it allows them to simplify radicals. And it's equally important for them to see that $\sqrt{a+b}$ is not $\sqrt{a} + \sqrt{b}$.

The final question is a trap, of course—many of them will answer x^4. But they should know they can test their answer by squaring back, and oops, $\left(x^4\right)^2$ is not x^{16}.

There are two ways to look at this problem correctly. One is that $\sqrt{x^{16}}$ asks a question: "What number, squared, is x^{16}? The rules of exponents are enough to answer this with x^8. The other way to look at it is to remember that raising something to the $\frac{1}{2}$ power is the same as taking a square root. So $\sqrt{x^{16}}$ is the same as $\left(x^{16}\right)^{\frac{1}{2}}$ which, again by the rules of exponents, is x^8. (I prefer the first way.)

Homework:

"Homework: Radicals"

[1] This content is available online at <http://cnx.org/content/m19484/1.2/>.

When going over the homework the next day, make sure to talk about the last few problems (the inverse functions). Make sure they tested them! There are several points that you want to make sure they got.

- x^2 has no perfect inverse. $\sqrt{x}$ works only if we confine ourselves to positive numbers. On the other hand, $\sqrt[3]{x}$ is a perfect inverse of $x3$. I always take a moment here to talk about what $\sqrt[3]{x}$ is, and also to make sure they understand that it is not the same thing as $3\sqrt{x}$, so it's vitally important to be careful in how you write it.
- x^3 and $3x$ are completely different functions. This is why exponents really have two different inverses, logs and radicals. Our last unit was on one of them, this unit will be on the other.

8.2 A Bunch of Other Stuff About Radicals[2]

Yeah, it's sort of a grab bag—a miscellaneous compilation of word problems, review from yesterday, and so on. You can just get them started working on the assignment after you're done going over yesterday's homework.

Toward the end, however, they are going to start running into trouble. This is when you introduce rationalizing the denominator. You may want to bring the whole class together to see who can figure out how to rationalize $\frac{1}{\sqrt{3}+1}$. It's a great opportunity to review our rules of multiplying binomials: $(a+b)^2 = a^2 + 2ab + b^2$ which is why multiplying by $\sqrt{3}+1$ doesn't work; $(a-b)^2 = a^2 - b^2$ which is why multiplying by $\sqrt{3}-1$ does.

But please be very careful here, because this particular topic has a very subtle danger. A lot of teachers communicate the idea that denominators should always be rationalized, "just because"—because I said so, or because somehow $\frac{\sqrt{2}}{2}$is "simpler" than $\frac{1}{\sqrt{2}}$. This is one of the best ways to convince students that math just doesn't make sense.

What I'm trying to do with this exercise is demonstrate a real practical benefit of rationalizing the denominator, which is that it helps you add and subtract fractions. It's difficult or impossible to come up with a common denominator without doing this first!

And of course, we have the "you never understand a function until you've graphed it" question. Talk a bit about the graph after they get it. They should be able to see that the domain and range are both ≥ 0 and why this must be so. They should see that for large values, it grows very slowly (like a log), but without the drastic behavior that the log shows in the $0 \leq x \leq 1$ range.

Homework:

"Homework: A Bunch of Other Stuff About Radicals"

8.3 Radical Equations[3]

If you read over the assignment carefully, I think it's pretty self-explanatory. Encourage the students to read it carefully as they go (not just skip to the equations).

In my experience, most the trouble with this section comes from trying to make easy problems, hard (that is, squaring both sides when you don't need to); or from trying to make hard problems, easy (neglecting to square both sides when you should). Make sure they are clear on the distinction—if there is a variable under the radical, you will need to square; otherwise, you won't.

Also, it can't hurt to say this about a hundred times: whenever you square both sides, you have to check your answers—they may not work even if you did all your math right! Make sure they understand when this rule applies, and also why squaring both sides can introduce false answers. (I work through that explanation pretty carefully in the "Conceptual Explanations.")

Homework:

"Homework: Radical Equations"

[2]This content is available online at <http://cnx.org/content/m19483/1.1/>.

[3]This content is available online at <http://cnx.org/content/m19485/1.2/>.

8.3.1 Time for another test!

Not much to say here, except that the real point of the extra credit is to see if they realize that the behavior will be very similar on the right, but it will extend down to the left as well.

Chapter 9

Imaginary Numbers

9.1 Introduction[1]

This is an interesting unit in several ways, both good and bad.

The good news is, it's fun. It's like a game, and I always try to present it that way.

The bad news is, it's incredibly abstract. It's abstract because it's hard to understand these numbers-that-aren't-numbers, and it's also abstract because, to save my life, I can't come up with any good explanation of why imaginary numbers are useful. Of course, they are useful—invaluable even—but how can I explain that to an Algebra II student? Here are a few things I do always say (several times).

1. These numbers are indeed useful, and they are used in the real world, even if I can't do a great job of explaining why to you right now.
2. Nothing in the real world is imaginary. That is, you will never have i tomatoes, or measure a brick that is $5i$ feet long, or wait for $3i+2$ seconds. So why are these useful? Because there are very often problems where the problem is real, and the answer is real, but in between, as you get from the problem to the answer, you have to use imaginary numbers. Repeat this several times. You have a problem, or real-world situation, which (of course) involves all real numbers. You do a bunch of math, which includes imaginary numbers. In the end, you wind up with the answer, which (of course) involves all real numbers again. But it would have been difficult or impossible to find that answer, if you didn't have imaginary numbers.
3. One example is electrical engineering. In an electric circuit you have resistors, capacitors, and inductors. They all act very differently in the circuit. When you model the circuit mathematically, to determine how it will behave (what current will flow through it), the inductor has an inductance and the capacitor has a capacitance and the resistor has a resistance and they are all very different, which makes the math really hairy. However, you can define a complex quantity called impedance which makes resistors, capacitors, and inductors all look mathematically the same. The disadvantage is that you are now working with a complex number instead of all real numbers. The advantage is that resistors, capacitors, and inductors now look the same in the equations, which makes life a whole lot simpler. So this is a good example of how you use complex numbers to make the math easier. (As a side note, electrical engineers call the imaginary number j whereas everyone else on the planet calls it i. I think kids like that bit of trivia.)
4. Imaginary numbers are also used in many other applications, such as quantum mechanics.

It's all very hand-wavy, and I admit that up front, and I don't hold the kids responsible for it. But I want them to know that this is really useful, and at the same time, I want to explain why we aren't going to have any "real world" problems in this unit.

[1]This content is available online at <http://cnx.org/content/m19424/1.2/>.

This lecture, by the way, usually comes toward the end of day 1, or during day 2—not at the very beginning of day 1. In the beginning, I prefer to treat it as a game—"What if there were a square root of –1? Just suppose, what if there were?" With one class I went so far in treating it as a game that it was day 3 before they realized I wasn't making the whole thing up.

Oh yeah, one more thing. The calculators will do imaginary numbers for them. I never tell them this. If they figure it out, more power to them. But I literally don't tell them until after the test that the calculator knows anything at all about imaginary numbers! I want them to be able to do these things on their own.

9.2 Introduction to Imaginary Numbers[2]

This is a fun day, or possibly two days. The first exercise is something that, in theory, they could walk all the way through on their own. But it sets up all the major themes in imaginary numbers.

In practice, of course, some groups will have problems, and will need help at various points. But beyond that, almost no groups will see the point of what they have done, even if they get it right. So a number of times in class, you are going to interrupt them and pull them back together into a classwide discussion, and discuss what they have just done. The ideal time to do this is after everyone in the class has reached a certain point—for instance, after they have all done #2 (or struggled with it in vain), you pull them back and talk about #2. All my suggested interruptions are described below.

Before you start, remind them that the equation $x^2 = - - 1$ has no answer, and talk about why. Then explain that we are going to pretend it has an answer. The answer is, of course, an "imaginary" number, so we will call it i. The definitions of i is therefore $i = \sqrt{-1}$ or, equivalently, $i^2 = - - 1$.

There are two ways to play this. One is to go into the whole "why i is useful" spiel that I spelled out above. The other approach, which is the one I take, is to treat it as a science fiction exercise. I always start by telling the class that in good science fiction, you start with some premise: "What if time travel were possible?" or "What if there were a man who could fly?" or something like that. Then you have to follow that premise rigorously, exploring all the ramifications of that one false assumption. So that is what we are going to do with our imaginary number. We are going to start with one false premise: "What if you could square something and get –1?" And we are going to follow that premise logically, using all the rules of math, and see where it would lead us.

Then they get started. And in #2, they get stopped in their tracks. So you give them a minute to struggle, and then walk it through on the board like this.

- $- - i$ means $- - 1 \bullet i$ (*Stress that this is not anything unusual about i, it is a characteristic of –1. We could just as easily say –2 means $- - 1 \bullet 2$, and so on. So we are treating i just like any other number.)
- So $i\,(- - i)$ is $i \bullet - - 1 \bullet i$.
- But we can rearrange that as $i \bullet i \bullet - - 1$. (You can always rearrange multiplication any way you want.)
- But $i \bullet i$ is –1, by definition. So we have $- - 1 \bullet - - 1$, so the final answer is 1.

The reason to walk through this is to get across the idea of what I meant about a science fiction exercise. Everything we just did was simply following the rules of math—except the last step, where we multiplied $i \bullet i$ and got –1. So it illustrates the basic way we are going to work: assume that all the rules of math work just like they always did, and that $i^2 = -1$.

The next few are similar. Many of them will successfully get $\sqrt{-25}$on their own. But you will have to point out what it means. So, after you are confident that they have all gotten past that problem (or gotten stuck on it), call the class back and talk a bit. Point out that we started out by just defining a square root of -1. But in doing so, we have actually found a way to take the square root of any negative number! There are two ways to see this answer. One is (since we just came off our unit on radicals) to write $\sqrt{-25} = \sqrt{25 \cdot -1} = \sqrt{-25}\sqrt{-1} = 5i$. The other—which I prefer—is to say, $\sqrt{-25}$ is asking the question "What number squared is -25? The answer is $5i$. How do you know? Try it! Square $5i$ and see what you get!"

[2]This content is available online at <http://cnx.org/content/m21990/1.1/>.

Now remind them of the subtle definition of square root as the positive answer. If you see the problem $x^2 = -25$ you should answer $x = \pm 5i$ (take a moment to make sure they all got the right answer to #4, so they see why $(-5i)^2$ gives -25). On the other hand, $\sqrt{-25}$ is just $5i$.

Next we move on to the cycle of powers. Again, they should be able to do this largely on their own. If a group needs a hint, remind them that if we made a similar table with powers of 2 (2^1, 2^2, 2^3, and so on), we would get from each term to the next one by multiplying by 2. So they should be able to figure out that in this case, you get from each term to the next by multiplying by i, and they should be able to do the multiplication. They will see for themselves that there is a cycle of fours. So then you can ask the whole class what i^{40} must be, and then i^{41} and so on, and get them to see the general algorithm of looking for the nearest power of 4. (I have tried mentioning that we are actually doing modulo 4 arithmetic and I have stopped doing this—it just confuses things. I do, however, generally mention that the powers of –1 go in a cycle of 2, alternating between 1 and –1, so this is just kind of like that.)

#13 is my favorite "gotcha" just to see who falls into the trap and says it's 9–16.

After they do #18, remind them that this is very analogous to the way we got square roots out of the denominator. And this is not a coincidence—i is a square root, after all, that we are getting out of the denominator! You may want to introduce the term "complex conjugate" even at this stage, but the real discussion of complex numbers will come later.

Homework:

"Homework: Imaginary Numbers"

When going over this homework the next day, make sure they got the point. Our "pattern of fours" can be walked backward as well as forward. It correctly predicts that $i^0 = 1$ which it should anyway, of course, since anything$^0 = 1$. It correctly predicts that $i^{-1} = -i$ which is less obvious—but remind them that, just yesterday, they showed in class that $\frac{1}{i}$ simplifies to $- - i$!

9.3 Complex Numbers[3]

The first thing you need to do is define a complex number. A complex number is a combination of real and imaginary numbers. It is written in the form $a + bi$, where a and b are both real numbers. Hence, there is a "real part" (a) and an "imaginary part" (bi). For instance, in $3 + 4i$, the real part is 3 and the imaginary part is $4i$.

At this point, I like to try to put this in context, by talking about all the different kinds of numbers we have seen. We started with counting numbers: 1, 2, 3, 4, and so on. If you are counting pebbles, these are the only numbers you will ever need.

Then you add zero, and negative numbers. Are negative numbers real things? Can they be the answers to real questions? Well, sure. . .depending on the question. If the question is "How many pebbles do you have?" or "How many feet long is this stick?" then the answer can never be –2: negative numbers are just not valid in these situations. But if the answer is "What is the temperature outside?" or "How much money is this company worth?" then the answer can be negative. This may seem like an obvious point, but I'm building up to something, so make sure it's clear—we have invented new numbers for certain situations, which are completely meaningless in other situations. I also stress that we have gone from the counting numbers to a more general set, the integers, which includes the counting numbers plus other stuff.

Then we add fractions, and the same thing applies. If the question is "How many pebbles do you have?" or "How many live cows are on this farm?" the answer can never be a fraction. But if the question is "How many feet long is this stick?" a fraction may be the answer. So again, we have a new set—the rational numbers—and our old set (integers) is a subset of it. And once again, these new numbers are meaningful for some real life questions and not for others. I always mention that "rational numbers" ("rational" not meaning "sane," but meaning rather a "ratio") are always expressable as the ratio of two integers, such as $\frac{1}{2}$ or $\frac{--22}{7}$. So since we have already defined the integers, we can use them to help define our larger set, the rational numbers.

[3]This content is available online at <http://cnx.org/content/m19423/1.4/>.

But some numbers are not rational—they cannot be expressed as the ratio of two integers. These are the irrational numbers. Examples are π, e, and $\sqrt{2}$ (or the square root of any other number that is not a perfect square).

If we add those to our collection—put the rational and irrationals together—we now have all the real numbers. You can draw a number line, going infinitely off in both directions, and that is a visual representation of the real numbers.

And now, finally, we have expanded our set even further, to the complex numbers, $a + bi$. Just as we piggybacked the definition of rational numbers on top of our definition of integers, we are piggybacking our definition of complex numbers on top of our definition of real numbers.

All that may sound unnecessary, and of course, it is. But some students really get into it. I have had students draw the whole thing into a big Venn diagram—which I did not ask them to do. (*It is a good extra credit assignment, though.) My own diagram is at the very end of this unit in the "Conceptual Explanations," under the heading "The World of Numbers." Many students like seeing all of math put into one big structure. And it helps make the point that complex numbers—just like each other generalization—are valid answers to some questions, but not to others. In other words, as I said before, you will never measure a brick that is 5i inches long. (It never hurts to keep saying this.)

The complex numbers are completely general—any number in the world can be expressed as $a + bi$. This is not obvious! There are plenty of things you can write that don't look like $a + bi$. One example is $\frac{1}{i}$ which does not look like $a + bi$ but can be put into that form, as we have already seen. Other examples are 2^i and $\ln(i)$, which we are not going to mess with, but they are worth pointing out as other examples of numbers that don't look like $a + bi$, but take-my-word-for-it you can make them if you want to. And then there is $\sqrt{i}$, which we are going to tackle tomorrow.

That is probably all the setup you need. They can do the in-class exercise on Complex Numbers and see for themselves that whether you add, subtract, multiply, or divide them, you get back to a complex number. Also make sure they get the point about what it means for two complex numbers to be equal: this will be very important as we move on. One other thing I like to mention at some point (doesn't have to be now) is that there are no inequalities with imaginary numbers. You cannot meaningfully say that $1 > i$ or that $1 < i$. Because they cannot be graphed on a number line, they don't really have "sizes"—they can be equal or not, but they cannot be greater than or less than each other.

Homework:

"Homework: Complex Numbers"

When going over this homework, make a special point of talking about #12. This helps reinforce the most important point of the year, about generalizations. Once you have found what happens to $(a + bi)$ when you multiply it by its complex conjugate, you have a general formula which can be used to multiply any complex number by its complex conjugate, without actually going through the work. Show them how #6, 8, and 10 can all be solved using this formula. Also, remind them that a and b are by definition real—so the answer $a^2 + b^2$ is also real. That is, whenever you multiply a number by its complex conjugate, you get a real answer. This is why there is no possible answer to #14.

9.4 Me, Myself, and the Square Root of i[4]

This is arguably the most advanced, difficult thing we do all year. But I like it because it contains absolutely nothing they haven't already done. It's not here because it's terribly important to know $\sqrt{i}$, or even because it's terribly important to know that all numbers can be written in $a + bi$ format. It is here because it reinforces certain skills—squaring out a binomial (always a good thing to practice), working with variables and numbers together, setting two complex numbers equal by setting the real part on the left equal to the real part on the right and ditto for the imaginary parts, and solving simultaneous equations.

Explain the problem we're going to solve, hand it out, and let them go. Hopefully, by the end of class, they have all reached the point where they know that $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $- - \frac{1}{\sqrt{2}} - - \frac{1}{\sqrt{2}}i$ are the two answers,

[4]This content is available online at <http://cnx.org/content/m19425/1.2/>.

and have tested them.

Note that right after this in the workbook comes a more advanced version of the same thing, where they find $\sqrt[3]{-1}$ (all three answers: $--1$, $\frac{1}{2}+\frac{\sqrt{3}}{2}i$, and $\frac{1}{2}--\frac{\sqrt{3}}{2}i$). I tried using this for the whole class, and it was just a bridge too far. But you could give it to some very advanced students—either as an alternative to the $\sqrt{i}$ exercise, or as an extra credit follow-up to it.

Homework:

They should finish the worksheet if they haven't done so, including #7. Then they should also do the "Homework: Quadratic Equations and Complex Numbers." It's a good opportunity to review quadratic equations, and to bring in something new! (It's also a pretty short homework.)

When going over the homework, make sure they did #3 by completing the square—again, it's just a good review, and they can see how the complex answers emerge either way you do it. #4 is back to the discriminant, of course: if $b^2-4ac<0$ then you will have two complex roots. The answer to #6 is no. The only way to have only one root is if that root is 0. (OK, 0 is technically complex…but that's obviously not what the question meant, right?)

The fun is seeing if anyone got #5. The answer, of course, is that the two roots are complex conjugates of each other—real part the same, imaginary part different sign. This is obvious if you rewrite the quadratic formula like this:

$$x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4\text{ac}}}{2a}$$

and realize that the part on the left is always real, and the part on the right is where you get your i from.

9.4.1 Time for another test!

Not much to say here, except that you may want to reuse this extra credit on your own test—if they learn it from the sample (by asking you) and then get it right on your test, they learned something valuable.

Chapter 10

Matrices

10.1 Introduction[1]

This is a "double" unit—that is, it is so long that I have a major test right in the middle of it.

The EOC spends an inordinate amount of time on problems like this:

The Kind of Problem I Don't Bother Too Much With

This matrix shows McDonald's sales for a three-day period.

	Big Macs	Fries	Coke
Monday	$1,000	$500	$2,000
Tuesday	$1,500	$700	$2,700
Wednesday	$800	$800	$1,500

Table 10.1

What were their total sales on Monday? What were their total sales of Big Macs? On which day did they make the most profit? etc etc...

I guess the object is to make it appear that "matrices are useful" but it is really deceptive. Of course, matrices are useful, but not because they give you a convenient way to organize tabular data and then add columns or look things up.

So I don't spend much time on this kind of thing. I start with what a matrix is (which is sort of like that). I develop the rules for adding matrices, subtracting them, multiplying a matrix by a constant, and setting two matrices equal to each other—all of which are very obvious, and should not be presented as a mystery, but rather just as something obvious.

Then comes the big two days of magic, in which we learn to multiply matrices. I use a "gradebook" application which gives an example of why you would want to do this strange operation—it makes a lot of sense up to the point where you are multiplying an arbitrary-dimensions matrix by a column matrix, although it gets a bit strained when you expand the second matrix. No matter. They need to get the mechanics of how you multiply matrices, and just practice them.

After that bit of magic, the rest should follow logically. The definition of $[I]$, the definition of an inverse matrix and how you find one, and (the final hoorah) the way you use matrices to solve linear equations, should all be logical and consistent, based on the one magic trick, which is multiplying them. Oh, also there is a magic trick where you find determinants, which doesn't have much to do with anything else.

There is one other thing I need to address, which is calculators. There is a day that I set aside to teach them explicitly how to do matrices on the calculator. But that day is after the first test. Before that day, I

[1]This content is available online at <http://cnx.org/content/m19445/1.1/>.

don't mention it at all. And even after that day, I stress doing things by hand, and give them problems that will force them to do so (by using variables). But I do love showing them that you can solve five equations with five unknowns quickly and easily by using matrices and a calculator!

10.1.1 Introduction to Matrices

Tell them to get into groups and work on "Introduction to Matrices." I think it is very self-explanatory. You may want to make the analogy at some point that setting two matrices equal to each other is kind of like setting two complex numbers equal to each other: for "this" to equal "that," all their respective parts must be equal.
Homework:
"Homework—Introduction to Matrices"

10.2 Multiplying Matrices I (row × column)[2]

Once again, you may just want to get them started on the assignment, and let it speak for itself.

However, toward the end of class—after they have all struggled through, or gotten stuck—pull them back and do some blackboard talk. Start by pointing out that we are doing two different things here. Multiplying a matrix times a constant is both easy and intuitive. If you can add two matrices, then you can add $[A]+[A]$ and thus figure out what $2\,[A]$ has to be.

On the other hand, multiplying two matrices (a row times a column) is not easy or intuitive. Show how to multiply a row matrix by a column matrix, do a few examples, and talk about how it applies to the gradebook example. In other words, make sure they get it.

What I do, about a hundred times, is try to get them to visualize the row floating up in the air and twisting around so that it lines up with the column. I use my hands, I use sticks, anything to get them to visually see this row floating up and twisting to line up with that column. This visualization is not essential today, but if they get it today, it will really help them tomorrow, when we start multiplying full matrices! As I mentioned earlier—this is a radical departure from my normal philosophy—I am more concerned that they get the mechanics here (how to do the multiplication) than any sort of logic behind it. They should be able to see, for instance, that if the row and column do not have the same number of elements, then the matrix multiplication is illegal.

Oh yeah, one more thing—I always stress that when you multiply two matrices, the product is a matrix. In the case of a row times a column, it is a 1×1 matrix, but it is still a matrix, not a number.
Homework
"Homework—Multiplying Matrices I"

10.3 Multiplying Matrices II (the full monty)[3]

This time, you're going to have to lecture. You are going to have to explain, on the board, how to multiply matrices. Probably a good 20 minutes (half the class) dedicated to showing them that this row goes over here to this column, and then we go down to the next row, and so on. Get them to work problems at their desks, make sure they are cool with it. You can also refer them to the "Conceptual Explanations" to see a problem worked out in a whole lot of detail.

Two things to stress:

1. Keep doing the visualization of a row (in the first matrix) floating up and twisting to get next to a column (in the second matrix). If the two do not line up—that is, they have different numbers of elements—then the multiplication is illegal.

[2]This content is available online at <http://cnx.org/content/m19448/1.1/>.
[3]This content is available online at <http://cnx.org/content/m19449/1.2/>.

2. Matrix multiplication does not commute. If you switch the order, you may turn a legal multiplication into an illegal one. Or, you may still have a legal multiplication, but with a different answer. AB and BA are completely different things with matrices.

You may never get to the in-class assignment at all. If you don't, that's OK, just skip it! However, note that the in-class assignment is built on one particular application, which is showing how Professor Snape can do just one matrix multiplication to get the final grades for all his students. This exercise is one of the few applications I have for matrix multiplication.

Homework:

"Homework—Multiplying Matrices II"

#4 is important for a couple of reasons. First, of course, by using variables, it forces them to do the work manually even if they have figured out how to do it on a calculator. More importantly, it continues to hammer home that message about what variables are—you can solve this leaving x, y, and z generic, and then you can plug in numbers for them if you want.

#5 and #7 set up the identity matrix; #6 sets up using matrices to solve linear equations. You don't need to mention any of that now, but you may want to refer back to them later. I don't want them to think of $[I]$ as being defined as "a diagonal row of 1s." I want them to know that it is defined by the property $\mathrm{AI} = \mathrm{AI} = A$, and to see how that definition leads to the diagonal row of 1s. #7 is the key to that.

10.4 Identity and Inverse Matrices[4]

This may, in fact, be two days masquerading as one—it depends on the class. They can work through the sheet on their own, but as you are circulating and helping, make sure they are really reading it, and getting the point! As I said earlier, they need to know that $[I]$ is defined by the property $\mathrm{AI} = \mathrm{IA} = A$, and to see how that definition leads to the diagonal row of 1s. They need to know that A^{-1} is defined by the property $\mathrm{AA}^{-1} = A^1 = I$, and to see how they can find the inverse of a matrix directly from this definition. That may all be too much for one day.

I also always mention that only a square matrix can have an $[I]$. The reason is that the definition requires I to work commutatively: AI and IA both have to give A. You can play around very quickly to find that a 2×3 matrix cannot possibly have an $[I]$ with this requirement. And of course, a non-square matrix has no inverse, since it has no $[I]$ and the inverse is defined in terms of $[I]$!

Homework:

"Homework—The Identity and Inverse Matrices"

10.5 Inverse of the Generic 2x2 Matrix[5]

This is one of those things that should be easy, but it isn't. It should be easy because they have already been doing it, with numbers, and it's just the same with letters. But hey, that's what Algebra II is about, right?

They should definitely work in groups here. Make sure they understand what they are doing. A clear sign that they don't understand what they are doing, even a little, is that they wind up solving for a, or solving for w in terms of x, or something like that. They need to understand that the object is to solve for w, x, y, and z in terms of a, b, c, and d. Only by doing this can they come up with a generic solution to the inverse of a 2×2 matrix, which can then be used quickly and easily to find the inverse of any 2×2 matrix. If they don't understand that, they just don't have any idea what we're doing—it's important to get them to understand the problem instead of just focusing on solving it.

[4]This content is available online at <http://cnx.org/content/m19443/1.1/>.
[5]This content is available online at <http://cnx.org/content/m19446/1.1/>.

The answer, by the way, is A-1= $\frac{1}{\text{ad}-\text{bc}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Once they have it, in this form, help them understand how to use it—the numbers in this diagonal switch places, the number in that diagonal change signs. This also helps set up the determinant (ad-bc), by the way.

10.6 Use Matrices for Transformation[6]

This is a fun day. No new math, just a cool application of the math we've seen.

The in-class assignment pretty well speaks for itself. It's worth mentioning that, despite the very simplified and silly nature of this specific assignment, the underlying message—that matrices are used to transform images in computer graphics—is absolutely true.

Some students (good students) may question why the last column $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is necessary in specifying Harpoona's initial condition. The reason is this representation is not simply a list of all the corner points in a shape. Yes, each column represents a point. But the matrix is a set of instructions to the computer, to draw lines from this point to that point. Without the last column, Harpoona would be missing her hypotenuse.

Homework

Homework: "Homework: Using Matrices for Transformation." Here we see a matrix that rotates any object by 300, counter-clockwise. The inverse matrix, of course, rotates clockwise. Hopefully they will discover all that for themselves. And hopefully most of them will realize why an inverse matrix always does the exact opposite of the original matrix, since if you do them one after the other, you end up back where you started.

10.6.1 Time for Another Test!

Our first test on matrices.

10.7 Matrices on the Calculator[7]

This starts with a lecture. You have to show them how to do matrices on the calculator. They should be able to. . .

- Enter several matrices at once
- Add several matrices
- Subtract
- Multiply
- Find inverses

All of this is explained step-by-step in the "Conceptual Explanations."

There are two things I stress. First, whenever I enter a matrix, I always check it. For instance, after I enter matrix [A], I go back to the home screen and go [A][Enter], and the calculator displays matrix [A] to me, so I can make sure I typed it right. One small mistype will ruin a whole problem, and it's really easy to do!

Second, the calculator is very smart about interpreting equations. After you enter three matrices, you can just type [A][B]-1-[C] and it will multiply [A] by the inverse of [B] and then subtract [C].

Homework

"Homework—Calculators." Depending on how things go, they may be able to finish this in class and have no homework.

[6]This content is available online at <http://cnx.org/content/m19451/1.1/>.
[7]This content is available online at <http://cnx.org/content/m19447/1.1/>.

10.8 Determinants[8]

Another very lecture-heavy topic, I'm afraid. Like multiplying matrices, finding the determinant is something you just have to show on the board. And once again, you can refer them in the end to the "Conceptual Explanations" to see an example worked out in detail.

Start by talking about the ad-bc that played such a prominent role in the inverse of a 2×2 matrix. This is, in fact, the determinant of a 2×2 matrix.

Then show them how to find the determinant of a three-by-three matrix, using either the "diagonals" or "expansion by minors" method, whichever you prefer. (I would not do both. Personally, I prefer "expansion by minors," and that is the one I demonstrate in the "Conceptual Explanations.")

Hot points to mention:

- Brackets like this [A] mean a matrix; brackets like this |A| mean a determinant. A determinant is a number associated with a matrix: it is not, itself, a matrix.
- Only square matrices have a determinant.
- Also show them how to find a determinant on the calculator. They need to be able to do this (like everything else) both manually and with a calculator.
- To find the area of a triangle whose vertices are (a,b), (c,d), and (e,f), you can use the formula: Area $= \frac{1}{2} \begin{vmatrix} a & c & e \\ b & d & f \\ 1 & 1 & 1 \end{vmatrix}$. This is the only use I can really give them for determinants. They will need to know this for the homework. Do an example or two. I like to challenge them to find that area any other way, just to make the point that it is not a trivial problem without matrices. (I don't know any other good way.)
- All we're really going to use "expansion by minors" for is 3×3 matrices. However, I like to point out that it can be obviously extended to 4×4, 5×5, etc. It also extends down to a 2×2—if you "expand minors" on that, you end up with the good old familiar formula ad-bc.
- Finally, mention that any matrix with determinant zero has no inverse. This is analogous to the rule that the number 0 is the only number with no inverse.

Homework
"Homework—Determinants"

10.9 Solving Linear Equations[9]

Just let them start this assignment, it should explain itself.

This is the coolest thing in our whole unit on matrices. It is also the most dangerous.

The cool thing is, you can solve real-world problems very quickly, thanks to matrices—so matrices (and matrix multiplication and the inverse matrix and so on) prove their worth. If you are given:

$2x - 5y + z = 1$
$6x - y + 2 = 4$
$4x - 10y + 2z = 2$

you plug into your calculator $[A] = \begin{bmatrix} 2 & -5 & 1 \\ 6 & -1 & 2 \\ 4 & -10 & 2 \end{bmatrix}$, $[B] = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$, ask for $A^{-1}B$, and the answer pops out! They should be able to do this process quickly and mechanically.

[8] This content is available online at <http://cnx.org/content/m19442/1.1/>.
[9] This content is available online at <http://cnx.org/content/m19450/1.1/>.

But the quick, mechanical nature of the process is also its great danger. I want them to see the logic of it. I want them to see exactly why the equation $\begin{bmatrix} 2 & -5 & 1 \\ 6 & -1 & 2 \\ 4 & -10 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ is exactly like those three separate equations up there. I want them to be able to solve like this:

$\mathrm{AX} = B$

$A^{-1}\mathrm{AX} = A^{-1}B$

$\mathrm{IX} = A^{-1}B$

$X = A^{-1}B$

to see why it comes out as it does. If they see that this is all perfectly logical, and they know how to do it, the day is a big success—and in fact, this sort of justifies the whole unit on matrices.

As a final point, mention what happens if the equations were unsolvable: matrix A will have a 0 determinant, and will therefore have no inverse, so the equation won't work. (You get an error on the calculator.)

Homework

"Homework—Solving Linear Equations"

10.9.1 Time for Another Test!

And, time to conclude our unit on matrices.

Chapter 11

Modeling Data with Functions

11.1 Introduction[1]

This unit is really three different topics, joined together by a somewhat weak thread.

The first topic is direct and inverse variation. The second topic is finding a parabola that fits any three given points. The third topic is regression on a calculator.

The weak thread that connects them is that they all involve starting with data, and finding a function that models that data. (At least, the latter topics are clearly about that, and the first topic is about that if you approach it the way I do...)

11.2 Direct and Inverse Variation[2]

This is one of those days where you want to get them working right away (on the "Direct Variation" assignment), let them finish the assignment, and then do 10-15 minutes of talking afterwards. You want to make sure they got the point of what they did.

Direct variation is, of course, just another kind of function—an independent variable, and a dependent variable, and consistency, and so on. But the aspect of direct variation that I always stress is that when the independent number doubles, the dependent number doubles. If one triples, the other triples. If one is cut in half, the other is cut in half...and so on. They are, in a word, proportional.

This is not the same thing as saying "When one goes up, the other goes up." Of course that is true whenever you have direct variation. But that statement is also true of ln(x), $\sqrt{x}$, x2, 2x, x+3, and a lot of other functions: they do "when one goes up, the other goes up" but not "when one doubles, the other doubles" so they are not direct variation. The only function that has that property is f(x)=kx, where k is any constant. (Point out that k could be $\frac{1}{2}$, or it could be –2, or any other constant—not just a positive integer.)

In #3 they arrived at this point. They should see that the equation y=kx has the property we want, because if you replace x with 2x then y=k(2x) which is the same thing as 2kx which is twice what it used to be. So if x doubles, y doubles.

They should also see that direct variation always graphs as a line. And not just any line, but a line through the origin.

But the thing I most want them to see is that there are many, many situations where things vary in this way. In other words, #4 is the most important problem on the assignment. You may want to ask them to tell the whole class what they came up with, and then you throw in a few more, just to make the point of how common this is. The amount of time you spend waiting in line varies directly with the number of people

[1]This content is available online at <http://cnx.org/content/m19454/1.2/>.
[2]This content is available online at <http://cnx.org/content/m19452/1.1/>.

in front of you; the amount you pay varies directly with the number of meals you order; the weight of your french fries measured in grams varies directly with the weight of your french fries measured in pounds; and both of these, in turn, vary directly with the number of fries; and so on, and so on, and so on.

11.2.1 Homework

Part I

"Homework: Inverse Variation"

That's right, no homework on direct variation—it's time to develop the second one. They should be able to do it pretty well on their own, in analogy to what happened in class. But you will spend a fair amount of the next day debriefing them on inverse variation, just as you did on direct. The defining property is that when the independent variable doubles, the dependent variable chops in half. Again, it is true to say "when one goes up, the other goes down"—but it is not enough. 1/x2 has that property, and so does 10-x, and neither of those is inverse variation.

Examples are a bit harder to think of, off the top of your head. But there is an easy and systematic way to find them. I always warn the students that I will ask for an example of inverse variation on the test, and then (now that I have their attention) I explain to them how to do it. Inverse variation is y=k/x (graphs as a hyperbola). This equation can be rewritten as xy=k. This is useful for two reasons. First, it gives you the ability to spot inverse variation—if the product is always roughly the same, it's inverse. Second, it gives you the ability to generate inverse variation problems, by thinking of any time that two things multiply to give a third thing, and then holding that third thing constant.

For instance: the number of test questions I have to grade is the number of students, times the number of questions on the test. That's obvious, right?

If I want to turn that into a direct variation problem, I hold one of the two multiplying variables constant. What do I mean "hold it constant?" I mean, pick a number. For instance, suppose there are twenty students in my class. Now the dependent variable (number of questions I have to grade) varies directly with the independent variable (number of questions I put on the test).

On the other hand, if I want to turn that exact same scenario into an inverse variation problem, I hold the big variable constant. For instance, suppose I know that I am only capable of grading 200 problems in a night. So I have to decide how many questions to put on the test, based on how many students I have. You see? Double the number of students, and the number of questions on the test drops in half.

Algebraically, if I call t the number of questions on the test, s the number of students, and g the number of questions I have to grade, then g=ts. In the first case, I set s=20 so I had the direct variation equation g=20t. In the second case I set g=200 so I had the inverse variation equation t=200/s.

I explain all that to my class, slowly and carefully. They need to know it because the actual test I give them will have a question where they have to make up an inverse variation problem, and I always tell them so. That question, if nothing else, will come as no surprise at all.

Part II

"Homework: Direct and Inverse Variation"

This is on the long side, and introduces a few new ideas: the idea of being proportional to the square (or square root) of a variable, and the idea of dependence on multiple variables. So you should ideally hand it out in the middle of class, so they have time to work on it before the homework—and be prepared to spend a lot of time going over it the next day.

11.3 Calculator Regression[3]

Lecture time. Talk about the importance of regression again. Walk them, step-by-step, through a few regressions on the calculator. (Detailed instructions for a TI-83 are included in the "Conceptual Explanations" for this section.)

My advice right now is to look at the points first (which may require resetting the window!) and then categorize them according to concavity—although I don't use that word. I talk about three kinds of increasing functions:

- Linear functions increase steadily

[3]This content is available online at <http://cnx.org/content/m19453/1.1/>.

- Logs and square roots increase more and more slowly
- Parabolas and exponential functions increase more and more quickly

Similarly, of course, for decreasing functions. Choose an appropriate kind of regression, and let the calculator do the rest.

After a few examples, hand out the homework.

Homework

"Homework: Calculator Regression"

11.3.1 Time for Another Test!

Question 11c is not something I would put on a real test, for a couple of reasons. First, it would be pretty hard to grade; second, and more importantly, it is unlike anything we've seen on a homework. But I might use it for an extra credit, and in any case, it can't hurt them to see it and discuss it on a sample.

Chapter 12

Conics

12.1 Introduction[1]

The last topic! And it's a big one. Here is the overall plan.

We start with a day on distance—finding the distance between two points, finding the distance from a point to a line—this will form a basis for the whole unit.

Then we do our shapes: circles, parabolas, ellipses, and hyperbolas. For each shape, there are really two things the students need to learn. One is the **geometric definition** of the shape. The other is what I call the **machinery** associated with the shape—the standard equation and what it represents. I cover these two things separately, and then connect them at the end.

12.2 Distance[2]

Before I hand this out, I tell them a bit about where we're going, in terms of the whole unit. We're going to do "analytic geometry"—that is, linking geometry with algebra. We're going to graph a bunch of shapes. And everything we're going to do is built upon one simple idea: the idea of **distance**.

Let's start with distance on a number line. (Draw a number line on the board.) Here's 4, and here's 10. What's the distance between them? Right, 6. You don't need to do any math, you can just count—1, 2, 3, 4, 5, 6. There we are.

How about the distance from 4 to 1? Right, 3. 4 to 0? Right, 4. One more: what is the distance from 4 to -5? Again, just count...1,2,3,4,5,6,7,8,9. The distance is nine.

So, what's going on here? In each case, we're counting from 4 to some other number: to 10, to 1, to 0, then to -5. What happened mathematically? We **subtracted**. This is a point that looks incredibly obvious, but really it isn't, so it's worth repeating: **if you subtract two numbers, you get the distance between them**. 10-4 is 6. 4-1 is 3. And the last one? $4 - (-5) = 4 + 5 = 9$. So even that works. Remember that we saw, by counting, that the distance from 4 to -5 is 9. Now we see that it works mathematically because subtracting a **negative** is like adding a **positive**.

Oh yeah...what if we had subtracted the other way? You know, 4-9 or -5-4. We would have gotten the right answers, only negative. But the distance would still be positive, because distance is always positive.

So, based on all that, at your seats, write down a **formula** for the distance from a to b on a number line: go! **(give them thirty seconds)** What did you get? $|a-b|$? Good! $|b-a|$? Also good! They are the same thing. *a-b* and *b-a* are **not** the same thing, but when you take the absolute value, then they are.

Now, let's get two-dimensional here. We'll start with the easy case, which is when the points line up. In that case, we can use the same rule, right? For instance, let's look at (4,3) and (10,3). How far apart are

[1]This content is available online at <http://cnx.org/content/m19307/1.1/>.

[2]This content is available online at <http://cnx.org/content/m19299/1.2/>.

they? Same as before—6. We can just count, or we can just subtract, because the y-coordinates are the same. **(Show them this visually!)** Similarly, suppose we take (-2,5) and (-2,-8). Since the x-coordinates are the same, we can just count again, or just subtract the y-coordinates, and get a distance of 13.

Now, what if **neither** coordinate is the same? Then it's a bit trickier. But we're not going to use any magic "distance formula"—if you ever memorized one, throw it out. All we need is what we've already done. Let's look at (-2,1) and (4,9). **(Draw it!)** To find that distance, we're going to find the distance **across** and the distance **up.** So draw in this other point at (4,1). Now, draw a triangle, with the distance we want over here, and the distance across here, and the distance up here. These two sides are easy, because they are just what we have already been doing, right? So this is 6 and this is 8. So how do we find this third side, which is the distance we wanted? Right, the Pythagorean Theorem! So it comes out as 10.

The moral of the story is—**whenever you need to find a distance, use the Pythagorean Theorem.**

OK, one more thing before you start the assignment. That was the distance between two **points.**How about the distance from **a point to a line?** For instance, what is the distance from you to the nearest street? The answer, of course, is—it depends on where on the street you want to get. But when we say, distance from you to the street, we mean the **shortest** distance. **(Do a few drawings to make sure they get the idea of shortest distance from a point to a line. If the line is vertical or horizontal, then we are back to just counting. If it's diagonal, life gets much more complicated, and we're not going to get into it. Except I sometimes assign, as an extra credit assignment, "find the distance from the arbitrary point** (x,y) **to the arbitrary line** $y = mx + b$. **It's ugly and difficult, but I usually have one or two kids take me up on it. For the rest of the class, just promise to stick with horizontal and vertical lines, and counting.)**

After all that is said, they are ready to start on the assignment.

Homework:

"Homework: Distance"

12.3 Circles[3]

Our first shape. (Sometimes I have started with parabolas first, but I think this is simpler.)

Here's how you're going to start—and this will be the same for every shape. **Don't tell them what the shape is.** Instead, tell them this. A bunch of points are getting together to form a very exclusive club. The membership requirement for the club is this: **you must be exactly 5 units away from the origin.** Any point that fulfills this requirement is in the club; any point that is either too close or too far, is not in the club. Give them a piece of graph paper, and have them draw all the points in the club. You come around and look at their work. If they are stuck, point out that there are a few very obvious points on the x-axis. The point is that, before any math happens at all, every individual group should have convinced themselves that this club forms a circle.

Then you step back and say—in Geometry class you used circles all the time, but you may never have formally defined what a circle is. We now have a formal definition: **a circle is all the points in a plane that are the same distance from a given point.** That distance is called what? (Someone will come up with "radius.") And that given point is called what? (Someone may or may not come up with "center.") The center plays a very interesting role in this story. It is the most important part, the only point that is key to the definition of the circle—but it is **not itself part of the circle,** not itself a member of the club. (The origin is not 5 units away from the origin.) This is worth stressing even though it's obvious, because it will help set up less obvious ideas later (such as the focus of a parabola).

Point out, also, that—as predicted—the definition of a circle is based entirely on the idea of **distance.** So in order to take our general geometric definition and turn it into math, we will need to mathematically understand distance. Which we do.

At this point, they should be ready for the in-class assignment. They may need some help here, but with just a little nudging, they should be able to see how we can take the **geometric definition** of a circle leads

[3]This content is available online at <http://cnx.org/content/m19298/1.2/>.

very directly to the **equation** of a circle, $(x-h)^2+(y-k)^2=r^2$ where (h,k) is the center and r is the radius. This formula should be in their notes, **etc.** Once we have this formula, we can use it to immediately graph things like $(x-2)^2+(y+3)^2=25$—to go from the equation to the graph, and vice-versa.

Once you have explained this and they get it, they are ready for the homework. They now know how to graph a circle in standard form, but what if the circle **doesn't** come in standard form? The answer is to complete the square—twice, once for x and once for y. If it works out that the homework gets done in class (this day or the next day), you may want to do a brief TAPPS exercise with my little completing-the-square demo in the middle of the homework. However, they should also be able to follow it on their own at home.

Homework:

"Homework: Circles"

It may take a fair amount of debriefing afterward, and even a few more practice problems, before you are confident that they "get" the circle thing. They need to be able to take an equation for a circle in non-standard form, **put** it in standard form, and then graph it. And they need to still see how that form comes directly from the Pythagorean Theorem and the definition of a circle.

There is one other fact that I always slip into the conversation somewhere, which is: how can you look at an equation, such as $2x^2+3x+2y^2+5y+7=0$, and even tell if it is a circle? Answer: it has both an x^2 and a y^2 term, and they have the **same coefficient.** If there is no x^2 or y^2 term, you have a line. If one exists but not the other, you have a parabola. By the time we're done with this unit, we will have filled out all the other possible cases, and I expect them to be able to look at any equation and recognize immediately what shape it will be.

12.4 Parabolas, Day 1[4]

Once again, when you start, **don't tell them** we're doing parabolas! Tell them we're going to create another club. This time the requirement for membership is: you must be exactly the same distance from the point (0,3) that you are from the line $y=-3$. For instance, the point (3,3) is not part of our club—it is 3 units away from (0,3) and **six** units away from $y=-3$.

Now, let them work in groups on "All the Points Equidistant from a Point and a Line" to see if they can find the shape from just that. If they need a hint, tell them there is one **extremely** obvious point, and two **somewhat** obvious points. After that they have to dink around.

When all or most groups have it, go through it on the blackboard, something like this. The extremely obvious point is the origin. The "somewhat" obvious points are (-6,3) and (6,3). Show why all those work.

Now, can any point **below the** x-axis work? Clearly not. Any point below the x-axis is "obviously" (meaning, after you show them for a minute) closer to the line, than to the point.

So, let's start working up from the origin. The origin was in the club. As we move up, we are getting **closer** to the point, and **farther away** from the line. So how can we maintain equality? The only way is to move **farther away** from the point, by moving out. In this way, you sketch in the parabola.

Now, you introduce the terminology. We're already old friends with the **vertex** of a parabola. This point up here is called the **focus.** This line down here is the **directrix.** The focus and directrix are kind of like the center of a circle, in the sense that they are central to the **definition** of what a parabola is, but they are not themselves **part** of the parabola. The vertex, on the other hand, **is** a part of the parabola, but is not a part of the definition.

The directrix, of course, is a horizontal line: but what if it isn't? What is the directrix is vertical? Then we have a **horizontal parabola.** Of course it isn't a function, but it's still a shape we can graph and talk about, and we have seen them a few times before. If you have time, work through $x=4y^2--8y$.

Homework:

"Homework: Vertical and Horizontal Parabolas"

[4]This content is available online at <http://cnx.org/content/m19315/1.2/>.

12.5 Parabolas, Day 2[5]

What good are parabolas? We've already seen some use for graphing parabolas, in terms of modeling certain kinds of behavior. If I throw a ball into the air, **not** straight up, its path through the air is a parabola. But here is another cool thing: if parallel lines come into a parabola, they all bounce to the focus.

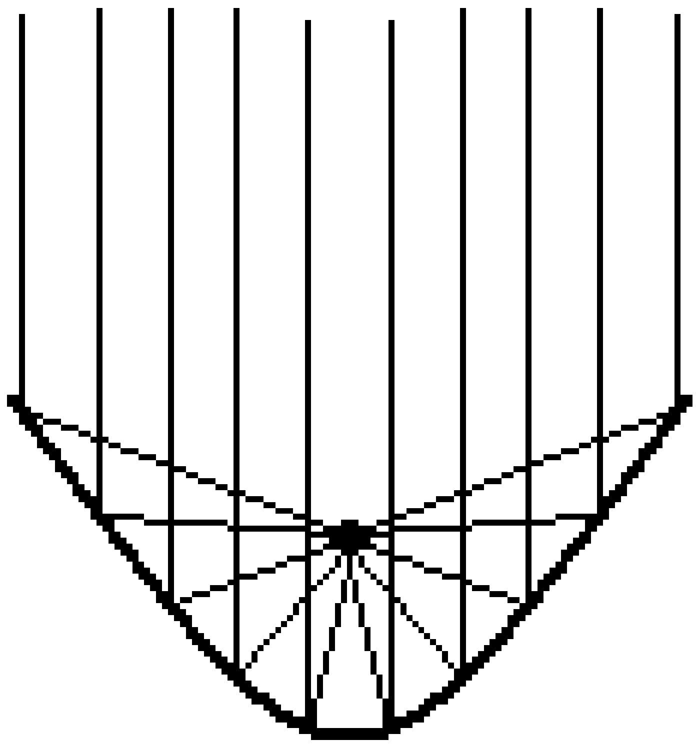

Figure 12.1: Draw this on the board:

So **telescopes** are made by creating **parabolic mirrors**—all the incoming light is concentrated at the focus. Pretty cool, huh?

We are already somewhat familiar with parabola machinery. We recall that the equation for a vertical parabola is $y = a(x-h)^2 + k$ and the equation for a horizontal parabola is $x = a(y-k)^2 + h$. The vertex in either case is at (h,k). We also recall that if a is positive, it opens up (vertical) or to the right (horizontal); if a is negative, it opens down or to the left. This is all old news.

Thus far, whenever we have graphed parabolas, we have found the vertex, determined whether they open up or down or right or left, and then gone "swoosh." If we were a little more sophisticated, we remembered that the width of the parabola was determined by a; so $y = 2x^2$ is narrower than $y = x^2$ which is narrower than $y = \frac{1}{2}x^2$. But how can we use a to actually draw the width accurately? That is the question for today. And the answer starts out with one fact which may seem quite unrelated: the distance from the vertex to the focus is $\frac{1}{4a}$. I'm going to present this as a magical fact for the moment—we will never prove it, though

[5]This content is available online at <http://cnx.org/content/m19313/1.2/>.

we will demonstrate it for a few examples. But that one little fact is the only thing I am going to ask you to “take my word for”—everything else is going to flow logically from that.

(Go back to your drawing of our parabola with focus (0,3) and directrix $y = -3$, and point out the distance from vertex to focus—label it $\frac{1}{4a}$.) Now, what is the distance from the vertex to the **directrix?** Someone should get this: it must also be $\frac{1}{4a}$. Why? Because the vertex is part of the parabola, so by definition, it must be the same distance from the directrix that it is from the focus. Label that.

Now, let's start at this point here (point to (-6,3)) and go down to the directrix. How long is that? Again, someone should get it: $\frac{1}{4a} + \frac{1}{4a}$, which is $\frac{2}{4a}$ or $\frac{1}{2a}$. Now, how far **over** is it, from this same point to the focus? Well, again, this is on the parabola, so it must be the same distance to the focus that it is to the directrix: $\frac{1}{2a}$.

If you extend that line all the way to the other side, you have two of those: $\frac{1}{2a} + \frac{1}{2a} = \frac{2}{2a}$, or $\frac{1}{a}$, running between (6,3) and (–6,3). This line is called the **latus rectum**—Latin for “straight line”—it is always a line that touches the parabola at two points, runs **parallel** to the directrix, and goes **through** the focus.

At this point, the blackboard looks something like this:

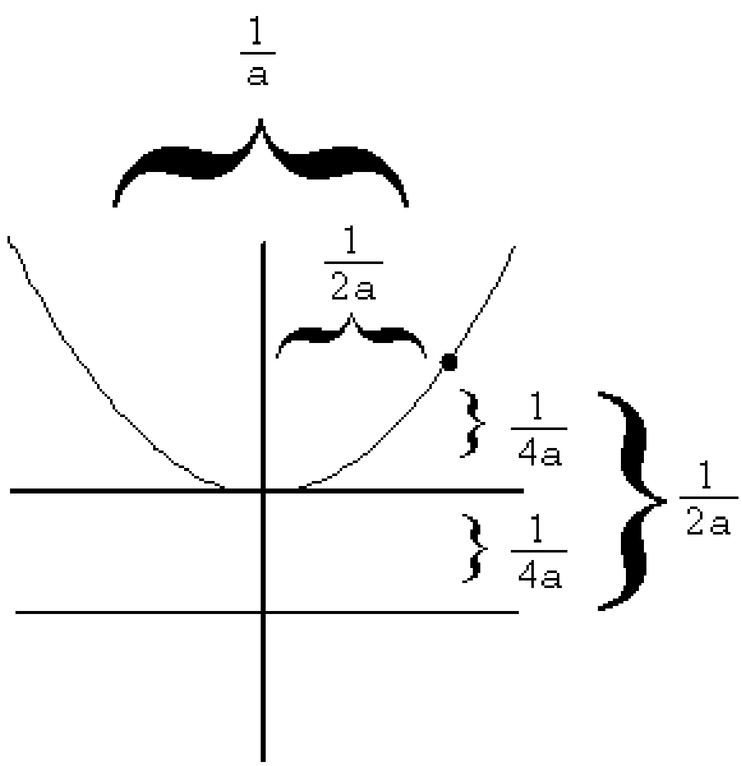

Figure 12.2

The point should be clear. Starting with the one magical fact that the distance from the focus to the vertex is $\frac{1}{4a}$, we can get everything else, including the length of the latus rectum, $\frac{1}{a}$. And that gives us a way of doing exactly what we wanted, which is using a to more accurately draw a parabola.

At this point, you do a sample problem on the board, soup to nuts. Say, $x = \frac{1}{2}(y+3)^2 - 4$. They should be able to see quickly that it is horizontal, opens to the right, with vertex at (-4,-3)—that's all review. But now we can also say that the distance from the focus to the vertex is $\frac{1}{4a}$ which in this case (since $a = \frac{1}{2}$) is $\frac{1}{2}$. So where is the focus? Draw the parabola quickly on the board. They should be able to see that the focus is to the **right** of the vertex, so that puts us at$(-3\frac{1}{2},-3)$. The directrix is to the **left,** at $x = -4\frac{1}{2}$. (I always warn them at this point, a very common way to miss points on the test is to say that the directrix is $(x = -4\frac{1}{2},-3)$. That's a point—the directrix is a line!)

How wide is the parabola? When we graphed them before, we had no way of determining that: but now we have the latus rectum to help us out. The length of the latus rectum is $\frac{1}{a}$ which is 2. So you can draw that in around the focus—going up one and down one—and then draw the parabola more accurately, staring from the focus and touching the latus rectum on both sides.

Whew! Got all that? Good then, you're ready for the homework! (You may want to mention that a different sample problem is worked in the "Conceptual Explanations.")

Homework:

"Homework: Parabolas and the Latus Rectum"

This one will take a lot of debriefing afterwards. Let's talk about #3 for a minute. The first thing they should have done is draw it—that can't be stressed enough—always draw it! Drawing the vertex and focus, they should be able to see that the parabola opens to the right. Since we know the vertex, we can write immediately $x = a(y-6)^2 + 5$. Many of them will have gotten that far, and then gotten stuck. Show them that the distance from the vertex to the focus is 2, and we know that this distance must be $\frac{1}{4a}$, so $\frac{1}{4a} = 2$. We can solve this to find $a = \frac{1}{8}$ which fills in the final piece of the puzzle.

On #5, again, before doing anything else, they should draw it! They should see that there is one vertical and one horizontal parabola that fits this definition.

But how can they find them? You do the vertical, they can do the horizontal. Knowing the vertex, we know that the vertical one will look like $y = a(x+2)^2 + 5$. How can we find a this time, by using the information that it "contains the point (0,1)"? Well, we have to go back to our last unit, when we found the equation for a parabola containing certain points! Remember, we said that if it **contains** a point, then that point must make the equation **true!** So we can plug in (0,1) to the equation, and solve for a. $1 = a(0+2)^2 + 5$, $4a + 5 = 1$, $a = -1$. The negative a tells us that the parabola will open down—and our drawing already told us that, so that's a good reality check.

12.6 Parabolas: From Definition to Equation[6]

OK, where are we? We started with the geometric definition of a parabola. Then we jumped straight to the machinery, and we never attempted to connect the two. But that is what we're going to do now.

Remember what we did with circles? We started with our geometric definition. We picked an arbitrary point on the circle, called it (x,y), and wrote an equation that said "you, Mr. (x, y), are exactly 5 units away from the origin." That equation became the equation for the circle.

Now we're going to do the same thing with a parabola. We're going to write an equation that says "you, Mr. (x, y), are the same distance from the focus that you are from the directrix." In doing so, we will write the equation for a parabola, **based on** the geometric definition. And we will discover, along the way, that the distance from the focus to the vertex really is $\frac{1}{4a}$.

That's really all the setup this assignment needs. But they will need a lot of help doing it. Let them go at it, in groups, and walk around and give hints when necessary. The answers we are looking for are:

1. $\sqrt{x^2 + y^2}$. As always, hint at this by pushing them toward the Pythagorean triangle.
2. $y + 4$. The way I always hint at this is by saying "Try numbers. Suppose instead of (x, y) this were (3,10). Now, how about (10,3)?" and so on, until they see that they are just adding 4 to the y-coordinate. Then remind them of the rule, from day one of this unit—to find distances, subtract. In this case, subtract -4.
3. $\sqrt{x^2 + y^2} = y + 4$. This is the key step! By asserting that $d1 = d2$, we are writing the definition equation for the parabola.

[6]This content is available online at <http://cnx.org/content/m19311/1.2/>.

4. Good algebra exercise! Square both sides, the y^2 terms cancel, and you're left with $x^2 = 8y + 16$. Solve for y, and you end up with $y = \frac{1}{8}x^2 - 2$. Some of them will have difficulty seeing that this is the final form—point out that we can rewrite it as $y = \frac{1}{8}(x-0)^2 - 2$, if that helps. So the vertex is (0,-2) and the distance from focus to vertex is 2, just as they should be.

Warn them that this **will be on the test!**

Which brings me to...

12.6.1 Time for another test!

Our first test on conics. I go back and forth as to whether I should give them a bunch of free information at the top of the test—but it's probably a good idea to give them a chart, sort of like the one on top of my sample.

12.7 Ellipses[7]

Only two shapes left! But these two are doozies. Expect to spend at least a couple of days on each—they get a major test all to themselves.

In terms of teaching order, both shapes are going to follow the same pattern that we set with parabolas. First, the geometry. Then, the machinery. And finally, at the end, the connection between the two.

So, as always, don't start by telling them the shape. Let them do the assignment "Distance to this point plus distance to that point is constant" in groups, and help them out until they get the shape themselves. A good hint is that there are two pretty easy points to find on the x-axis, and two harder points to find on the y-axis. As always, keep wandering and hinting until most groups have drawn something like an ellipse. Then you lecture.

The lecture starts by pointing out what we have. We have two points, called the **foci.** (One "focus," two "foci.") They are the defining points of the ellipse, but they are not part of the ellipse. And we also have a distance, which is part of the definition.

Because the foci were horizontally across from each other, we have a horizontal ellipse. If they were vertically lined up, we would have a vertical ellipse. You can also do diagonal ellipses, but we're not going to do that here.

Let's talk more about the geometry. One way you can draw a circle is to thumbtack a piece of string to a piece of cardboard, and tie the other end of the string to a pen. Keeping the string taut, you pull all the way around, and you end up with a circle. Note how you are using the geometric definition of a circle, to draw one: the thumbtack is the center, and the piece of string is the radius.

Now that we have our geometric definition of an ellipse, can anyone think of a way to draw one of those? **(probably not)** Here's what you do. Take a piece of string, and thumbtack **both** ends down in a piece of cardboard, so that the string is **not** taut. Then, using your pen, pull the string taut.

[7]This content is available online at <http://cnx.org/content/m19303/1.2/>.

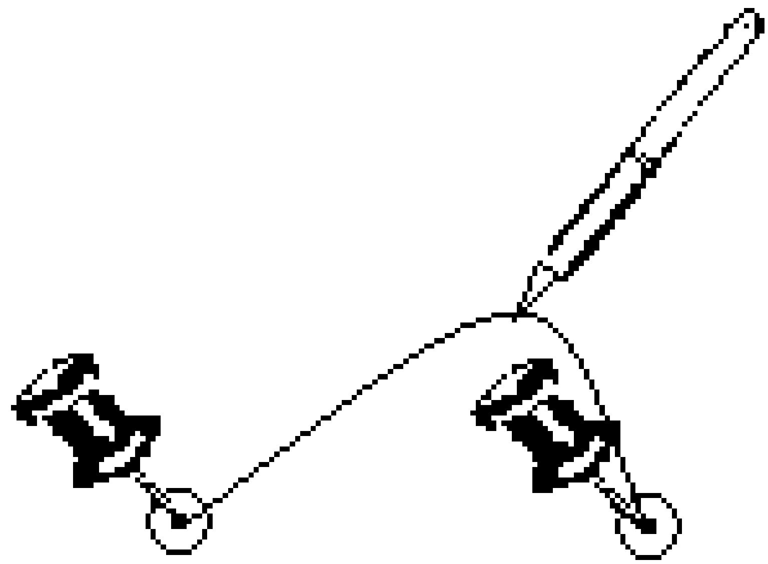

Figure 12.3

Now, pull the pen around, keeping the string taut. You see what this does? While the string is taut, the **distance from the pen to the left thumbtack, plus the distance from the pen to the right thumbtack, is always a constant**—namely, the length of the string. So this gives you an ellipse. I think most people can picture this if they close their eyes. Sometimes I assign them to do this at home.

OK, so, what good are ellipses? The best example I have is orbits. The Earth, for instance, is traveling in an **ellipse,** with the sun at one of the two foci. The moon's orbit around the Earth, or even a satellite's orbit around the Earth, are all ellipses.

Another cool ellipse thing, which a lot of people have seen in a museum, is that if you are in an elliptical room, and one person stands at each focus, you can hear each other whisper. Just as a parabola collects all incoming parallel lines at the focus, an ellipse bounces everything from one focus straight to the other focus.

OK, on to the machinery. Here is the equation for a horizontal ellipse, centered at the origin.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{12.1}$$

Here is a drawing of a horizontal ellipse.

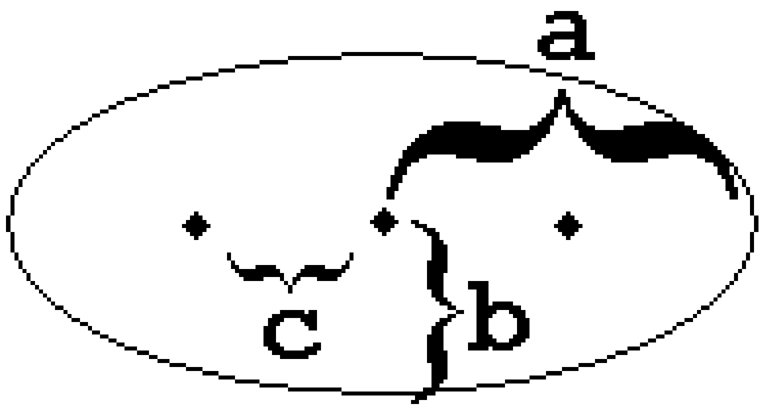

Figure 12.4

There are three numbers in this drawing. a is the distance from the center to the far edge (right or left). If you double this, you get the horizontal length of the entire ellipse—this length, $2a$ of course, is called the **major** axis.

b is the distance from the center to the top or bottom. If you double this, you get the vertical length of the entire ellipse—this length, $2b$ of course, is called the **minor** axis.

c is the distance from the center to either focus.

a and b appear in our equation. c does not. However, the three numbers bear the following relationship to each other: $a^2 = b^2 + c^2$. Note also that a is always the biggest of the three!

A few more points. If the center is **not** at the origin—if it is at, say, (h, k), what do you think that does to our equation? They should all be able to guess that we replace x^2 with $(x-h)^2$ and y^2 with $(y-k)^2$.

Second, a vertical ellipse looks the same, but with a and b reversed:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad (12.2)$$

Let's see how all that looks in an actual problem—walk through this on the blackboard to demonstrate. Suppose we want to graph this:

$x^2 + 9y^2 - -4x + 54y + 49 = 0$

First, how can we recognize it as an ellipse? Because it has both an x^2 and a y^2 term, and the coefficient are **different.** So we complete the square twice, sort of like we did with circles. But with circles, we always divided by the coefficient right away—this time, we pull it out from the x and y parts of the equation **separately.**

$x^2 + 9y^2 - -4x + 54y + 49 = 0$		**The original problem**
$x^2 - -4x + 9y^2 + 54y = -49$		**Group the x and y parts**
		continued on next page

$\left(x^2 - -4x\right) + 9\left(y^2 + 6y\right) = -49$		**Factor out the coefficients. In this case, there is no x coefficient, so we just have to do y . In general, we must do both.**
$\left(x^2 - -4x + 4\right) + 9\left(y^2 + 6y + 9\right) = -49 + 4 + 81$		**Complete the square, twice**
$(x-2)^2 + 9(y+3)^2 = 36$		**Finish completing that square**
$\frac{(x-2)^2}{36} + \frac{(y+3)^2}{4} = 1$		**Divide by 36. This is because we need a 1 on the right, to be in our standard form!**

Table 12.1

OK, get all that? Now, what do we have?

First of all, what is the center? That's easy: (2,-3).

Now, here is a harder question: does it open vertically, or horizontally? That is, does this look like $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or like $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$? The way to tell is by remembering that a is always bigger than b. So in this case, the 36 must be a^2 and the 4 must be b^2, so it is horizontal. We can see that the major axis ($2a$) will be 12 long, stretching from (-4,-3) to (8,-3). (Draw all this as you're doing it.) And the minor axis will be 4 long, stretching from (2,-5) to(2,-1).

And where are the foci? Since this is a horizontal ellipse, they are to the left and right of the center. By how much? By c. What is c? Well, $a^2 = b^2 + c^2$. So $c^2 = 32$, and $c = \sqrt{32} = 4\sqrt{2}$, or somewhere around $5\frac{1}{2}$. (They should be able to do that without a calculator: 32 is somewhere between 25 and 36, so $\sqrt{32}$ is around $5\frac{1}{2}$.) So the foci are at more or less ($-3\frac{1}{2}$,-3) and ($7\frac{1}{2}$,-3). We're done!

Homework:

"Homework: Ellipses"

There are two things here that may throw them for a loop.

One is the fractions in the denominator, and the necessity of (for instance) turning $25y^2$ into $\frac{y^2}{1/25}$. You may have to explain very carefully **why** we do that (standard form allows for a number on the bottom but not on the top), and **how** we do that.

The other is number 7. Some students will quickly and carelessly assume that 94.5 is a and 91.4 is b. So you want to draw this very carefully on the board when going over the homework. Show them where the 94.5 and 91.4 are, and remind them of where a, b, and c are. Get them to see from the drawing that $94.5 + 91.4$ is the major axis, and is therefore $2a$. And that $a - 91.4$ is c, so we can find c, and finally, we can use $a^2 = b^2 + c^2$ to find b. This is a really hard problem, but it's worth taking a lot of time on, because it really drives home the importance of visually being able to see an ellipse in your head, and knowing where a, b, and c are in that picture.

Oh, yeah…number 6 may confuse some of them too. Remind them again to **draw it first,** and that they can **plug in** (0,0) **and get a true equation.**

OK, now we're at a bit of a fork in the road. The next step is to connect the geometry of the ellipse, with the machinery. This is a really great problem, because it brings together a lot of ideas, including some of the work we did forever ago in radical equations. It is also really hard. So you can decide what to do based on how much time you have left, and how well you think they are following you. You may want to have them go through the exercise in class (expect to take a day). Or, you may want to make photocopies of the completely-worked-out version (which I have thoughtfully included here in the teacher's guide), and have them look it over as a TAPPS exercise, or just ask them to look it over. Or you could skip this entirely, or make it an extra credit.

12.8 Ellipses: From Definition to Equation[8]

[8] This content is available online at <http://cnx.org/content/m19305/1.1/>.

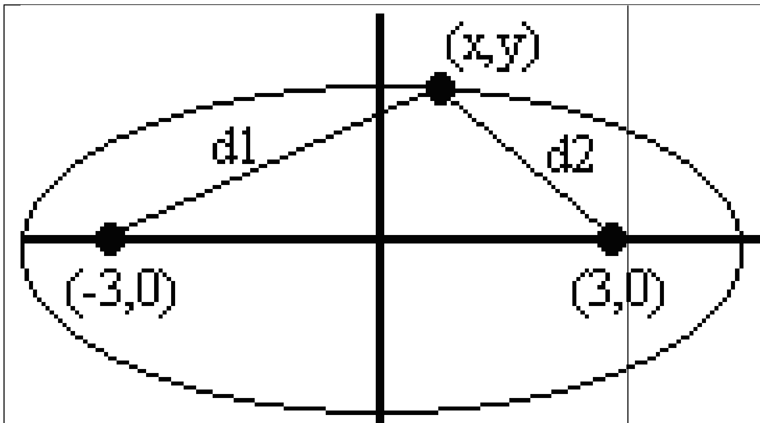	**Here is the geometric definition of an ellipse. There are two points called the "foci": in this case, (-3,0) and (3,0) . A point is on the ellipse if the sum of its distances to both foci is a certain constant: in this case, I'll use 10 . Note that the foci define the ellipse, but are not part of it.**

Table 12.2

The point (x, y) represents any point on the ellipse. $d1$ is its distance from the first focus, and $d2$ to the second. So the ellipse is **defined geometrically** by the relationship: $d1 + d2 = 10$.

To calculate $d1$ and $d2$, we use the Pythagorean Theorem as always: drop a straight line down from (x, y) to create the right triangles. Please verify this result for yourself! You should find that $d1 = \sqrt{(x+3)^2 + y^2}$ and $d2 = \sqrt{(x-3)^2 + y^2}$. So the equation becomes:

$\sqrt{(x+3)^2 + y^2} + \sqrt{(x-3)^2 + y^2} = 10$. **This defines our ellipse**

The goal now is to simplify it. We did problems like this earlier in the year (radical equations, the "harder" variety that have two radicals). The way you do it is by isolating the square root, and then squaring both sides. In this case, there are two square roots, so we will need to go through that process twice.

$\sqrt{(x+3)^2 + y^2} = 10 - -\sqrt{(x-3)^2 + y^2}$	**Isolate a radical**
$(x+3)^2 + y^2 = 100 - -20\sqrt{(x-3)^2 + y^2} + (x - -3)^2 + y^2$	**Square both sides**
$(x^2 + 6x + 9) + y^2 = 100 - -20\sqrt{(x-3)^2 + y^2} + (x^2 - -6x + 9) + y^2$	**Multiply out the squares**
$12x = 100 - -20\sqrt{(x-3)^2 + y^2}$	**Cancel & combine like terms**
$\sqrt{(x-3)^2 + y^2} = 5 - -\frac{3}{5}x$	**Rearrange, divide by 20**
	continued on next page

$(x - -3)^2 + y^2 = 25 - -6x + \frac{9}{25}x^2$	**Square both sides again**
$\left(x^2 - -6x + 9\right) + y^2 = 25 - -6x + \frac{9}{25}x^2$	**Multiply out the square**
$\frac{16}{25}x^2 + y^2 = 16$	**Combine like terms**
$\frac{x^2}{25} + \frac{y^2}{16} = 1$	**Divide by 16**

Table 12.3

...and we're done! Now, according to the "machinery" of ellipses, what should that equation look like? Horizontal or vertical? Where should the center be? What are a, b, and c? Does all that match the picture we started with?

12.9 Hyperbolas[9]

The good news about hyperbolas is, they are a lot like ellipses—a lot of what has already been learned, will come in handy here. The bad news about hyperbolas is, they are a lot like ellipses—so all the little differences can be very confusing.

We will start as always with the geometric definition. Let them do the assignment "Distance to this point **minus** distance to that point is constant" in groups, and help them out until they get the shape themselves. There are two pretty easy points to find on the x-axis, but from there they just sort of have to noodle around like we did with parabolas, asking...what happens as I move inside? What happens as I move outside? As always, keep wandering and hinting until most groups have drawn something like a hyperbola. Then you lecture.

The lecture starts by pointing out what we have. We have two points, once again called the **foci.** They are the defining points of the hyperbola, but they are not part of the hyperbola. And we also once again have a distance which is part of the definition.

Because the foci were horizontally across from each other, we have a horizontal hyperbola. If they were vertically lined up, we would have a vertical hyperbola. You can also do diagonal hyperbolas—anyone remember where we have seen one of those? That's right, inverse variation! That was a hyperbola, just like these. But we're not going to talk about those in this unit, just the horizontal and vertical ones.

Incidentally, a hyperbola is **not** two back-to-back parabolas. It looks like it, but these shapes are actually different from parabolic shapes, as we will see.

OK, so, what good are hyperbolas? The analogy continues...orbits! Suppose a comet is heading toward the sun. (Draw.) If it has a low energy—that is, a low velocity—it gets trapped by the sun, and wins up orbiting around the sun in an elliptical orbit. But if it has high energy (high velocity) it zooms around the sun and then zooms away forever. Its path in this case is half a hyperbola.

Another cool use is in submarine detection. A submarine sends out a pulse. Two receiving stations get the pulse. They don't know what direction it came from or when it was sent, but they do know that one station received it exactly two seconds before the other one. This enables them to say that the distance from the sub to this station, **minus** the distance to this other station, is such-and-such. And this, in turn, locates the sub on a hyperbola.

OK, on to the machinery. Here is the equation for a horizontal hyperbola, centered at the origin.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (12.3)$$

Looks familiar, doesn't it? But the plus has changed to a minus, and that makes all the difference in the world.

Here is a drawing of a horizontal hyperbola.

[9] This content is available online at <http://cnx.org/content/m19306/1.2/>.

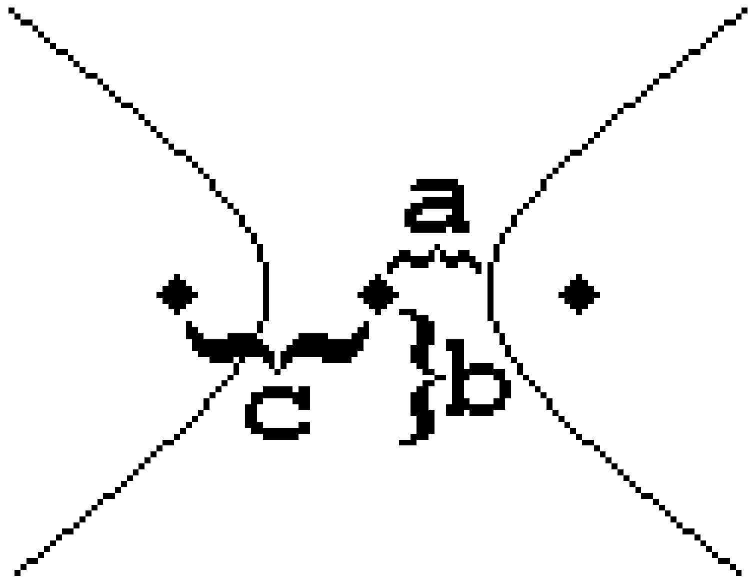

Figure 12.5

Let's be very careful in seeing how this **is,** and **is not,** like an ellipse.

a is defined in a very similar way. It goes from the center, to the edges. In this case, the edges are called the "vertices" (in analogy to parabolas). The distance from one vertex to the other ($2a$ of course) is called the **transverse axis.**

c is defined in a very similar way: it goes from the center to either focus.

b is perpendicular to the other two, just as before. But it goes from the center to...well, to a sort of strange point in the middle of nowhere. We're going to use this point. The distance from the top point to the bottom point ($2b$) is called the **conjugate axis.**

But here is one major difference. In an ellipse, the foci are **inside;** in a hyperbola, they are **outside.** So you can see, just looking at an ellipse, that $a > c$; and you can see, just looking at a hyperbola, that $c > a$. Hence, our equation relating the three shapes is going to be different. Instead of $a^2 = b^2 + c^2$, we have $c^2 = a^2 + b^2$. This reflects the fact that c is the biggest one in this case.

Once again, the class should be able to see that if the center is (h, k) instead of the origin, we replace x^2 with $(x-h)^2$ and y^2 with $(y-k)^2$.

How about a vertical hyperbola? That looks like this:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad (12.4)$$

The way we tell vertical from horizontal is **completely different.** In an ellipse, we told by assuming that $a > b$. In a hyperbola, we have no such guarantee; either a or b could be the greater, or they could even be the same. Instead, we look at which one comes **first.** If we are doing an $x^2 - y^2$ thing, it's horizontal; if we are doing a $y^2 - x^2$ thing, it's vertical. **This is a very common source of errors...be careful!**

There should be no need to go through the whole completing-the-square rigmarole on the board—just tell them it is exactly like with ellipses, including making sure you have a 1 on the right, and there is nothing multiplied by x^2 or y^2. But the really different part is the graphing. So let's just pick it up there. Suppose you want to graph:

$\frac{(x-2)^2}{36} - \frac{(y+3)^2}{4} = 1$

First of all, what is the center? That's easy: (2,-3).

Now, here is a harder question: does it open vertically, or horizontally? We can answer this question without even looking at the numbers on the bottom! The x^2 is positive and the y^2 is negative, so this is horizontal.

As we did with ellipses, we will then find a, b, and c. $a = 6$ and $b = 2$. We will find c with the hyperbola equation $c^2 = a^2 + b^2$ (different from the ellipse equation!) and get $c = \sqrt{40}$ which is $2\sqrt{10}$ or somewhere just above 6 (again, because 40 is just above 36).

Now, it's drawing time. We start at the center. We go out horizontally by 6 to find the vertices, and by a little more than 6 to find the foci. We go out vertically by 2 to find the endpoints of the conjugate axis, those weird little points in space. Then what?

Here's what you do. Draw a rectangle, going through the vertices, and the endpoints of the conjugate axis. Then, draw diagonal lines through the corners of that rectangle. Those diagonal lines are going to serve as **asymptotes,** or guides: they are not part of the hyperbola, but they help us draw it. Why? Because as it moves out, the hyperbola gets closer and closer to the asymptotes, but never quite reaches them. So once you have drawn your asymptotes, you have a guide for drawing in your hyperbola.

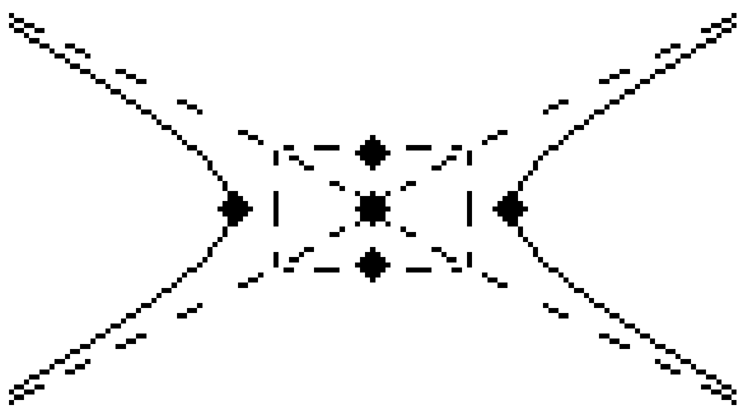

Figure 12.6

Note that I draw the box and the asymptotes in dotted lines, indicating that they are not really part of the hyperbola.

It's worth talking for a while about what an asymptote is, since it is such an important concept in Calculus—the line that the curve gets closer and closer and closer to, without ever quite reaching. It's also worth pointing out that this shows that a hyperbola is not two back-to-back parabolas, since parabolas do not display asymptotic behavior.

Finally, I always mention the comet again. Remember that if a comet comes in with high energy, it swoops around the sun and then flies away again. Now, if there were no sun—if there were nothing in the universe but our comet—the comet would travel in a straight line. And clearly, when the comet is very, very far away from the sun (either **before** or **after** its journey through our solar system), the effect of the sun is very small, so the comet travels almost in a straight line. That straight line is the asymptote. The farther away from the sun the comet gets, the closer it gets to that straight line.

Homework:
"Homework: Hyperbolas"

The only really unusual thing here is that I ask for the equation of one of the asymptotes. This is just a quick review of the skill of finding the equation for a line, given that you already know two points on the line.

You will note that I have absolutely nothing here about going from the geometry of the hyperbola, to the equation. The reason is that it is exactly like the ellipse. You may want to do it, or you may not. If you do it, you shouldn't need to hand them anything—just say "By analogy to what we did with the ellipse, do this."

Now that you have done all the shapes, the one vital skill that cuts across all of it is looking at an equation ($ax^2 + by^2 + cx + dy + e = 0$) and telling what shape it is. This is done entirely by looking at the coefficients of the squared terms (a and b), and you should refer them to the chart at the end of the "Conceptual Explanations."

12.9.1 Time for our very last test!

With luck, you have two or three weeks left for review after this, before the final test. Congratulations, you made it through!

Chapter 13

Sequences and Series

13.1 Prerequisites[1]

The major "prerequisite" for this unit is the introductory unit on **functions**. However, the introduction to geometric sequences will work best if the unit on **exponents** has already been covered; and the inductive proofs often require skills covered in the unit on **rational expressions**.

13.2 Arithmetic and Geometric Sequences[2]

The in-class assignment does not need any introduction. Most of them will get the numbers, but they may need help with the last row, with the letters.

After this assignment, however, there is a fair bit of talking to do. They have all the concepts; now we have to dump a lot of words on them.

A "sequence" is a list of numbers. In principal, it could be anything: the phone number 8,6,7,5,3,0,9 is a sequence.

Of course, we will not be focusing on random sequences like that one. Our sequences will usually be expressed by a formula: for instance, "the xxxnth terms of this sequence is given by the formula $100+3(n-1)$" (or $3n+97$ in the case of the first problem on the worksheet. This is a lot like expressing the **function** $y=100+3(x-1)$, but it is not exactly the same. In the function $y=3x+97$, the variable x can be literally any number. But in a **sequence**, xxxn must be a positive integer; you do not have a "minus third term" or a "two-and-a-halfth term."

The first term in the sequence is referred to as t_1 and so on. So in our first example, $t_5=112$.

The number of terms in a sequence, or the particular term you want, is often designated by the letter n.

Our first sequence adds the same amount every time. This is called an **arithmetic sequence**. The amount it goes up by is called the **common difference** d (since it is the difference between any two adjacent terms). Note the relationship to linear functions, and slope.

Exercise 13.2.1

If I want to know all about a given arithmetic sequence, what do I need to know? Answer: I need to know t_1 and d.

Exercise 13.2.2

OK, so if I **have** t_1 and d for the arithmetic sequence, give me a formula for the n^{th} term in the sequence. (Answer: $t_n=t_1+d(n-1)$. Talk through this carefully before proceeding.)

Time for some more words. A **recursive definition** of a sequence defines each term in terms of the previous. For an arithmetic sequence, the recursive definition is $t_{n+1}=t_n+d$. (For instance, in our

[1] This content is available online at <http://cnx.org/content/m19495/1.1/>.
[2] This content is available online at <http://cnx.org/content/m19490/1.1/>.

example, $t_{n+1} = t_n + 3$. An **explicit definition** defines each term as an absolute formula, like the $3n + 97$ or the more general $t_n = t_1 + d\,(n-1)$ we came up with.

Our second sequence multiplies by the same amount every time. This is called a **geometric sequence**. The amount it multiplies by is called the **common ratio** r (since it is the ratio of any two adjacent terms).

Exercise 13.2.3

Find the recursive definition of a geometric sequence. (Answer: $t_{n+1} = rt_n$. They will do the explicit definition in the homework.)

Exercise 13.2.4

Question: How do you make an arithmetic sequence go **down**? Answer: $d < 0$

Exercise 13.2.5

Question: How do you make a geometric series go down? Answer: $0 < r < 1$. (Negative r values get weird and interesting in their own way...why?)

Homework

"Homework: Arithmetic and Geometric Sequences"

13.3 Series and Series Notation[3]

Begin by defining a series: it's like a sequence, but with plusses instead of commas. So our phone number example of a sequence, "8,6,7,5,3,0,9" becomes the series "$8 + 6 + 7 + 5 + 3 + 0 + 9$" which is 38.

Many of the other words stay the same. The first term is t_1, the n^{th} term is t_n, and so on. If you add up all the terms of an arithmetic sequence, that's called an arithmetic series; and similarly for geometric.

The hardest part about this introduction is the notation. Explain about series notation, using weird examples like $\sum_{n=3}^{7} \frac{n^2-2}{5}$ just to make the point that even when the function looks complicated, it is not hard to **write out the terms**.

Note that the "counter" always goes by ones. Does this mean you can't have a series that goes up by 2s? Ask them how to use series notation for the series $10 + 12 + 14 + 16$.

Homework:

"Homework—Series and Series Notation"

13.4 Arithmetic and Geometric Series[4]

Going over the homework, make sure to mention #3(e), an **alternating series**. You get that kind of alternation by throwing in a $(-1)^n$ or, in this case, $(-1)^{n-1}$.

Last night's homework ended with the series "all the even numbers between 50 and 100." Some students may have written $\sum_{n=1}^{26} (48 + 2n)$. Others may have written the answer differently. But one thing they probably all agree on is that adding it up would be a pain. If only there were...a shortcut!

Let's consider the series $3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$. (Write that on the board.)

Exercise 13.4.1

What do we get if we add the first term to the last? Answer: 20. Modify your drawing on the board to look like this:

[3]This content is available online at <http://cnx.org/content/m19491/1.1/>.
[4]This content is available online at <http://cnx.org/content/m19494/1.1/>.

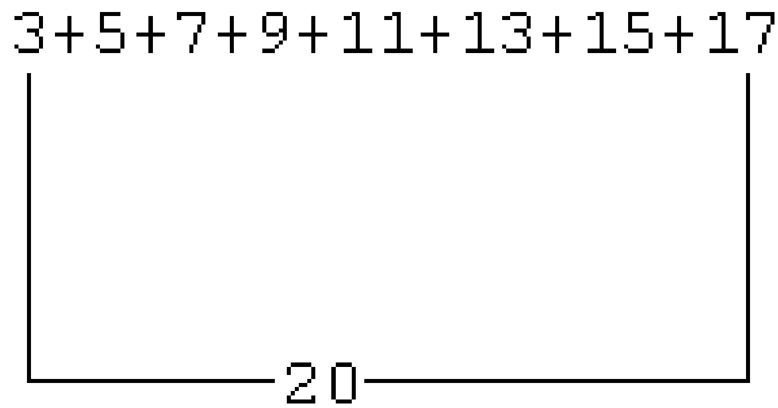

Figure 13.1

OK, what about the second term to the second-to-last? Hmm....20 again. Add to the drawing, and then keep adding until it looks like this:

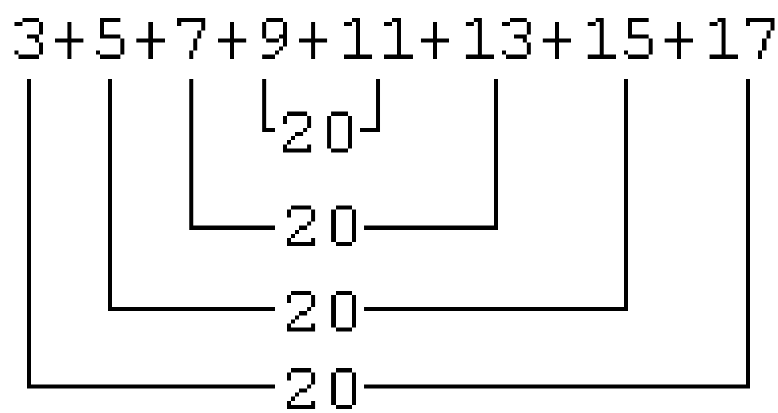

Figure 13.2

Exercise 13.4.2

So, looking at that drawing, what does $3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$ add up to? Hopefully everyone can see that it adds up to four 20s, or 80.

Exercise 13.4.3

And this is the big one—will that trick work for all series? If so, why? If not, which series will it work for? Answer: It will work for all **arithmetic series**. The reason that the second pair added up the same as the first pair was that we went up by two on the left, and down by two on the right. As long as you go up by the same as you go down, the sum will stay the same—and this is just what happens for arithmetic series.

OK, what about geometric series? Write the following on the board:

$2 + 6 + 18 + 54 + 162 + 486 + 1458$

Clearly the "arithmetic series trick" will not work here: $2 + 1458$ is not $6 + 486$. We need a whole new trick. Here it comes. First, to the left of your equation, write $S =$ so the board looks like:

$S = 2 + 6 + 18 + 54 + 162 + 486 + 1458$

where S is the mystery sum we're looking for. Now, **above** that, write:

$3S =$

ask the class what comes next. Can we just multiply each term by 3? (Yes, distributive property.) When you write this line, line up the numbers like this:

$3S = 6 + 18 + 54 + 162 + 486 + 1458 + 437$

$S = 2 + 6 + 18 + 54 + 162 + 486 + 1458$

But don't go too fast on that step—make sure they see why, if S is what we said, then $3S$ must be that!

Now, underline the second equation (as I did above), and then **subtract the two equations**. What do we get on the left of the equal sign? What do we get on the right? See how things cancel? See if you can get the class to tell you that...

$2S = 4374 - 2$

So then S is just 2186. They may want to verify this one on their calculators. Once again, however, the key is to understand why this trick **always works** for any Geometric series.

13.4.1 Homework:

"Homework: Arithmetic and Geometric Series"

13.5 Proof by Induction[5]

13.5.1 Proof by Induction

Going over last night's homework, make sure they got the right formulas. For an arithmetic series, $S_n = \frac{n}{2}(t_1 + t_n)$. For a geometric, $S_n = \frac{t_1 r^n - t_1}{r-1}$, in some form or another.

Example 13.1
Students sometimes ask if that formula will still work for an arithmetic series with an odd number of terms. Obviously, you can't still pair them up in the way we have been doing. The answer is, it does still work. One proof—which I usually don't mention unless the right questions are asked—is that, for an arithmetic series, the average of all the terms is right in the middle of the first and last terms. (It can take a minute to convince yourself that this is not always true for any series, but it is for any **arithmetic** series.) So the average is $\left(\frac{t_1+t_n}{2}\right)$, and there are n terms. This leads us to the total sum being $n\left(\frac{t_1+t_n}{2}\right)$, which is the old formula written in a new way. This lacks some of the elegance of the original proof, but it has the advantage that it doesn't matter if n is even or odd.

Anyway, on to today's topic.

NOTE: Important note. The following lecture can be done in 5-10 minutes—I've done it many times—and if you do it that way, it doesn't work. This can be one of the most confusing topics in the whole unit. It must be taken **very slowly and carefully**!

Today, we're going to learn a new way of proving things. This method, called "proof by induction," is a very powerful and general technique that turns up in many different areas of mathematics: we are going to be applying it to series, but the real point is to learn the technique itself.

So...to begin with, we are going to prove something we already know to be true:

$1 + 2 + 3 + 4 ... n = \frac{n}{2}(n+1)$

Of course we know how to prove that using the arithmetic series trick, but we're going to prove it a **different** way.

Let's start by seeing if that formula works when $n = 1$: in other words, for a 1-term series. In that case, what is the left side of the equation? (Even this seemingly innocuous question can baffle good students sometimes. Give them a minute. Point to the equation. Remind them that the equal sign divides any equation into a left side, and a right side. So, what is the left side of this equation, when there is only one term?) Yes, it is just...1.

How about the right side? Well, that's... $\frac{1}{2}(1+1) = 1$. So at least, for this particular case, it works.

To build up to the next step, ask this hypothetical question. Suppose we had not yet proven that this equation **always** works. But suppose that I had proven that it works **when** $n = 200$. Just say, I had sat down with my calculator and added up all the numbers from 1 to 200, which took a very long time, but in the end, I did indeed get what the formula predicts (which is, of course, $100 \times 201 = 20,100$. And now I ask you to confirm that the formula works when $n = 201$.

Well, you can do the right side easily enough: $\frac{201}{2}(202) = 20,301$. But what about the left side? Do you have to add up all those numbers on **your** calculator? No, you don't, if you're clever. (See if they can figure this next part out—this is the key.) I already told you what the first 200 numbers add up to. So you can simply add 201 to my total. $20,100 + 201 = 20,301$.

The point here is not just "it works." The point is that you can confirm that it works, **without adding up all 200 numbers again**, because I already did that part—all you have to add is the last number.

[5] This content is available online at <http://cnx.org/content/m19492/1.1/>.

Now...suppose I had already proven that it works for $n = 325$. How would we show that it works for $n = 326$? Good—we would add 326 to the old answer (for $n = 325$), and see if we got what the formula predicted we **should** get for $n = 326$. Let's try it...

Now...suppose I had already proven that it works for $n = 1000$. How would we show that it works for $n = 1001$? Good—we would add 1001 to the old answer (for $n = 1000$), and see if we got what the formula predicted we **should** get for $n = 1001$. Let's try it... *Repeat this exercise until they are sick of it, but boy, do they get it. Then hit them with the big one:* **what is the general form of this question***? See if they can figure out that it is:*

Suppose I had already proven that it works for some n. How would we show that it works for $n + 1$?

Give them time here...see if they can find the answer...

We would add $(n + 1)$ to the old answer (for n, and see if we got what the formula predicted we **should** get for $(n + 1)$.

What does that look like? Well, for the old n, the formula predicted we would get $\frac{n}{2}(n + 1)$. So if we add $(n + 1)$ to that, we get $\frac{n}{2}(n + 1) + (n + 1)$. And what **should** we get? Well, for $(n + 1)$, the formula predicts we should get $\frac{n+1}{2}(n + 1 + 1)$.

Do the algebra to show that they are equal. Then, step back and say...so, what have we done? Well, first we proved that the formula works for $n = 1$. Then we proved—not for one specific case, but quite generally—that **if it works for any number, it must also work for the next number**. If it works for $n = 1$, then it must work for $n = 2$. If it works for $n = 2$, then it must work for $n = 3$....and so on. It must always work.

At this point, I think it's helpful to work through one more example. I recommend going through $\Sigma\frac{1}{n(n+1)}$, just as it is done in the Conceptual Explanations. This time, you're using a little less explanation and focusing more on the process, so it makes a better model for their homework.

Homework:

"Homework—Proof by Induction"

At this point, you're ready for the test. Unlike most of my "Sample Tests," this one is probably too **short**, but it serves to illustrate the sorts of problems you will want to ask, and to remind the students of what we've covered.

13.6 Extra Credit[6]

13.6.1 An extra cool problem you may want to use as an extra credit or something

Exercise 13.6.1 *(Solution on p. 108.)*

A bank gives $i\%$ interest, compounded annually. (For instance, if $i = 6$, that means 6% interest.) You put A dollars in the bank **every year for** n **years**. At the end of that time, how much money do you have?

NOTE: (The fine print: Let's say you make your deposit on January 1 every year, and then you check your account on December 31 of the last year. So if $n = 1$, you put money in exactly once, and it grows for exactly one year.)

The **previous** year's money receives interest twice, so it is worth $A\left(1 + \frac{i}{100}\right)^2$ at the end. And so on, back to the first year, which is worth $A\left(1 + \frac{i}{100}\right)^n$ (since that initial contribution has received interest n times).

So we have a Geometric series:

$S = A\left(1 + \frac{i}{100}\right) + A\left(1 + \frac{i}{100}\right)^2 + ... + A\left(1 + \frac{i}{100}\right)^n$

We resolve it using the standard trick for such series: multiply the equation by the common ratio, and then subtract the two equations.

$\left(1 + \frac{i}{100}\right)S = A\left(1 + \frac{i}{100}\right)^2 + ... + A\left(1 + \frac{i}{100}\right)^n + A\left(1 + \frac{i}{100}\right)^{n+1}$

[6]This content is available online at <http://cnx.org/content/m19493/1.1/>.

$S = A\left(1 + \frac{i}{100}\right) + A\left(1 + \frac{i}{100}\right)^2 + ... + A\left(1 + \frac{i}{100}\right)^n$
$\left(\frac{i}{100}\right) S = A\left(1 + \frac{i}{100}\right)^{n+1} - -A\left(1 + \frac{i}{100}\right)$
$S = \frac{100A}{i}\left[\left(1 + \frac{i}{100}\right)^{n+1} - \left(1 + \frac{i}{100}\right)\right]$

Example 13.2

Example: If you invest \$5,000 per year at 6% interest for 30 years, you end up with:

$\frac{100(5000)}{6}\left[1.0631 - -1.06\right] = \$419,008.39$

Not bad for a total investment of \$150,000!

Solutions to Exercises in Chapter 13

Solution to Exercise 13.6.1 (p. 106)

The money you put in the very **last** year receives interest exactly once. "Receiving interest" in a year always means being multiplied by $\left(1+\frac{i}{100}\right)$. (For instance, if you make 6% interest, your money multiplies by 1.06.) So the A dollars that you put in the last year is worth, in the end, $A\left(1+\frac{i}{100}\right)$.

Chapter 14

Probability

14.1 Tree Diagrams[1]

You can start them off here with the in-class assignment "How Many Groups?" or even hand it out after the previous test: it isn't long or difficult, and it does not require any introduction.

It does, however, require a lot of follow-up. The worksheet leads to a lecture, and the lecture goes something like this.

The big lesson for the first day is how to make a chart of all the possibilities for these kinds of scenarios. For the die-and-coin problem, the tree diagram looks like (draw this on the board):

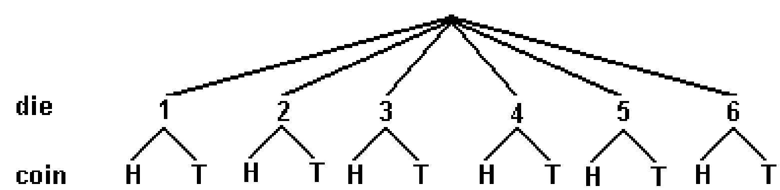

Figure 14.1

It may seem silly to repeat "Heads-Tails, Heads-Tails," and so on, six times. But if you do so, each "leaf" of this "tree" represents exactly one possibility. For instance, the third leaf represents "The die rolls a 2, and the coin gets heads." That's a completely different outcome from the fifth leaf, "The die rolls a 3 and the coin gets heads."

Using a tree like this, we can answer probability questions.

Exercise 14.1.1

What is the probability of the outcome "Die rolls 2, coin gets heads?" Just ask this question, give them 15 seconds or so to think about it, then call on someone for the answer. But then, talk through the following process for getting the answer.

1. Count the number of leaves that have this particular outcome. In this case, only one leaf.
2. Count the total number of leaves. In this case, twelve.
3. Divide. The probability is $\frac{1}{12}$.

[1] This content is available online at <http://cnx.org/content/m19463/1.1/>.

Exercise 14.1.2

What is the probability of the outcome "Die rolls a prime number, coin gets tails?" Give them 30 seconds or so, then go through it carefully on the chart. There are three such leaves. (*Trivia fact: 1 is not considered a prime number.) So the probability is $\frac{3}{12}$, or $\frac{1}{4}$.

Exercise 14.1.3

What does that really mean? I mean, either you're going to get that outcome, or you're not. After you roll-and-flip, does it really mean anything to say "The probability of that outcome was $\frac{1}{4}$?"

Give the class a minute, in pairs, to come up with the best possible explanation they can of what that statement, "The probability of this event is $\frac{1}{4}$," really means. Call on a few. Ultimately, you want to get to this: it doesn't really mean much, for one particular experiment. But if you repeat the experiment 1,000 times, you should expect to get this result about 250 of them.

Exercise 14.1.4 *(Solution on p. 119.)*

Why are there twelve leaves? (Or: How could you figure out that there are twelve leaves, without counting them?)

Get the idea? OK, let's try a new one.

A Ford dealer has three kinds of sedans: Ford Focus, Taurus, or Fusion. The Focus sedan comes in three types: S, SE, or SES. The Fusion sedan also comes in three types: S, SE, or SEL. The Taurus comes in only two models: SEL, and Limited. (All this is more or less true, as far as I can make out from their Web site.)

Ask everyone in the class, in pairs, to draw the appropriate tree and use it to answer the following questions.

- If you choose a car at random from the dealer lot, what are the odds that it is a Fusion? (Answer: $\frac{3}{8}$.)
- What are the odds that it is a Fusion SE? (Answer: $\frac{1}{8}$.)
- What are the odds that it is any kind of SEL? (Answer: $\frac{1}{4}$.)
- Finally—and most important—what assumption must you make in answering all of the above questions, that was not stated in the original description of the situation?

Hopefully someone will come up with the answer I'm looking for to that last question: you're assuming that the dealer's lot has exactly the same number of each possible kind of car. In real life, of course, that assumption is very likely wrong.

This ties in, of course, to the last question on the assignment they did. Red-haired people are considerably more rare than the other types. So this business of "counting leaves" only works when each leaf is exactly as common, or probable, as each other leaf. This does not mean we cannot do probability in more complicated situations, but it means we will need to develop a more sophisticated rule.

14.1.1 Homework

"Homework: Tree Diagrams"

When going over this homework the next day, there are two things you want to emphasize about problem #2 (stars). First: because the actual tree diagram would have 70 leaves, you don't want to physically draw it. You sort of have to imagine it. They have seen three diagrams now: the coin-and-die, the cars, and the three-coins. That should be enough for them to start to imagine them without always having to draw them.

Second: the answer to question 2(c) is not "one in 70, so maybe 14 or so." That logic worked fine with 1(e), but in this case, it is not reasonable to assume that all types of stars are equally common. The right answer is, "I can't answer this question without knowing more about the distribution of types."

Problem #3(e) is subtler. The fact that $\frac{1}{4}$ of the population is children does not mean that $\frac{1}{4}$ of the white population is children. It is quite possible that different ethnic groups have different age breakdowns. But ignoring that for the moment, problem #3 really brings out a lot of the main points that you want to make the next day:

14.2 Introduction to Probability[2]

OK, let's really talk about problem #3 from last night's homework. And for the moment, let's ignore part (e), and go ahead and assume that $\frac{1}{4}$ of all white people are children. Based on that assumption, you would expect roughly 75 white people, about 19 of whom are children, and about 9 of whom are boys. If you answered exactly $\frac{75}{8}$, or 9.375, that isn't a crazy answer. Of course, you can't actually have 9.375 white boys in a room. But that is actually the "expectation value" for such an experiment. If you have a thousand rooms with a hundred people each, the average number of white boys in each room will probably be 9.375.

In the formal language of probability, we would say that for any randomly chosen person in the U.S. in 2006, there is a 9.375% chance that this person will be a white boy. That's what "percent" means: out of a hundred.

Exercise 14.2.1

What percent of the people in this class, right now, are girls?

Exercise 14.2.2

If you roll a die, what is the percent chance that you will get an even number?

OK, that's easy enough. But we're going to tweak it a bit. Obviously, there is nothing magical about the number 100. We could just as easily ask "How many out of a thousand?" or "How many out of 365?" But what turns out to be most convenient, mathematically, is to ask the question "How many out of 1?"

This is how we are going to work with probability numbers from here on out, so it is very important to understand this numbering system!

- The probability of any event whatsoever, under any and all circumstances, is always between 0 and 1. A probability of –2, or a probability of 2, is meaningless.
- A probability of 0 means "It cannot possibly happen."
- A probability of 1 means "It is guaranteed to happen."
- A probability of $\frac{1}{4}$ means "It has a one in four chance of happening," or "If you try this 100 times, it will probably happen 25 of them," so it is the same as a 25% chance.

After you have said all that, you're ready to hit them with the worksheet "Introduction to Probability." It should only take 10 minutes (half of which is spent on #2a).

Then come back. Let's go over #2 carefully.

2b. The probability is $\frac{1}{16}$. You can see this from the tree diagram, but how could we have figured it out without a drawing? The answer—we've discussed this before, and it is absolutely central—is by multiplying. 4 possibilities for the first die, times 4 possibilities for the second die, makes 16 possibilities for the combination.

[2]This content is available online at <http://cnx.org/content/m19461/1.1/>.

But here's another way we can look at that same multiplication. The probability of "3 on the first die" is $\frac{1}{4}$. The probability of "2 on the second die" is also $\frac{1}{4}$. So the probability of both these events happening is $\frac{1}{4} \times \frac{1}{4}$, or $\frac{1}{16}$.

When you have two different independent events—that is, neither one has an effect on the other—the probability of both happening is the probability of the first one, times the probability of the second one.

The idea of "independent" events is crucial here, of course, and you have to stress it. But it's also a fairly obvious point, and there is a real danger of making it sound more esoteric than it is. If you spend ten minutes discussing the word "independent" you may do more harm than good. Consider trying this instead. Tell that class that you're looking into a big box full of bananas. One out of every four bananas in the box is green; the rest are yellow. Also, one out of every three bananas is stamped "Ship to California"; the rest say "ship to New York." Finally, half the bananas are over four days old.

- What is the probability that a given banana is green, and destined for New York? $\frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$. One out of every six bananas have both of these attributes. Or, to put it another way, a given randomly chosen banana has a $\frac{1}{6}$ chance of having both attributes.
- What is the probability that a given banana is green, and over four days old? Well, not much. Not $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$. Because in general, as a banana gets older, it turns from green to yellow. So being green, and being old, are not independent: one makes the other less likely.

Now, ask the class, in pairs, to come up with a similar scenario. (It should not involve fruit!) They should think of two events that are independent, and calculate the probability of both of them happening. Then they should think of two events that are not independent, and explain why the probability of both of them happening is not the product of their individual probabilities.

Homework

"Homework: The Multiplication Rule"

Going over this homework, of course you want to make sure that the last problem gets answered. With a little thought, it should be obvious to anyone that if P is the probability that something will occur, $1 - P$ is the probability that it will not occur. If it happens 1 time out of 5, then it doesn't happen 4 times out of 5. This can be memorized as a new rule, along with the multiplication rule, but it is easier to see why it works.

14.3 Trickier Probability Problems[3]

The thing that makes probability problems the darling of math contest writers everywhere, is also the thing that makes them frustrating for so many students: no two problems are exactly alike. Most probability problems can be solved with the multiplication rule, combined with a lot of good, hard thinking about the problem.

I'm going to present two scenarios with five questions here, in the lesson plan. The idea is for you to talk them through with the class. In each case, explain the scenario and the question clearly. Then give them a minute or two, with no guidance, to think about it. Then take their answers and go over the correct answer very slowly and clearly. None of them should be presented as if it were a symbol of a whole, unique, important class of problems. Each should be presented as simply another example of you can solve a wide variety of problems, if you're willing to think about them patiently and clearly.

Example 14.1: Scenario 1

You reach your hand into a bag of Scrabble® tiles. The bag has one tile with each letter. You pull out, first one tile, and then another.

1. What is the probability that you will pull out, first the letter A, and then the letter B? The quick, easy answer is $\frac{1}{26} \times \frac{1}{26}$. Quick, easy....and not quite right. Yes, there is a $\frac{1}{26}$ chance that the first tile will be an A. But once you have that tile, there are only 25 left. So the

[3]This content is available online at <http://cnx.org/content/m19464/1.2/>.

probability of the second tile being a B are actually $\frac{1}{25}$. The probability of getting an A followed by a B are $\frac{1}{26} \times \frac{1}{25}$.

2. What is the probability that your two tiles are the letters A and B? It looks like the same question, but there is a subtle difference. You could pull out A followed by B (as in the last example), or you could pull out B followed by A. So there are really two ways to do it, and the probability is $\frac{1}{25} \times \frac{1}{26} \times 2$.

Example 14.2: Scenario 2

You roll two 6-sided dice.

1. What is the probability that the sum of the two dice is 10? Imagine making a tree diagram. It would have 36 leaves. How many of them would have a sum of 10? 6–4, 5–5, and 4–6. (Of course, on the tree diagram, "6 on the first die, 4 on the second" is a different leaf from "4 on the first die, 6 on the second"...just as in the AB problem above.) So the probability is $\frac{3}{36}$, or $\frac{1}{12}$.
2. What is the probability that neither die rolls a 1? We do not have a "neither" rule, so we have to reframe the question in terms of the rules we do have. We can rephrase the question like this: what is the probability that the first die doesn't roll a 1, and the second die also doesn't roll a 1? The first is $\frac{5}{6}$, and the second is also $\frac{5}{6}$. So the probability of both happening is $\frac{25}{36}$. It's an easy question to answer, once you reword it correctly.
3. What is the probability that either die (or "at least one die") rolls a 1? (Most people think the answer will be $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. By that logic, by the time you roll six dice, you are guaranteed to get at least one 1: obviously not true!)

The right way to think about this problem is as the reverse, the "not," of the previous problem. We said that 25 out of 36 times, neither die will roll a 1. So the remaining 11 out of 36 times, at least one of them will. This is an example of the "not" rule we got from last night's homework: the probability of "no ones" is $\frac{25}{36}$, so the probability of NOT "no ones" is $1 - \frac{25}{36} = \frac{11}{36}$. (*It's interesting to note that the "naïve" guess of $\frac{1}{3}$ is not too far off, and makes a reasonable approximation. If you have a 1 in 10 chance of doing something, and you try three times, there is a roughly $\frac{3}{10}$ chance that you will succeed at least once—but not exactly $\frac{3}{10}$.)

The last thing you need to assure the class, before you hit them with the worksheet, is that no one is born knowing how to do this. Probability problems are just like everything else: they make more sense, and get easier, with practice. It's OK to get frustrated, but don't give up!

Then give them the worksheet. Ideally they should be able to make a good (10-15 minute) start in class, and then finish it up for homework. Expect to spend a lot of the next day going over these. It's worth it.

Homework

"Homework: Trickier Probability Problems"

14.4 Permutations[4]

No in-class worksheet today—a day of lecture.

How many different three-digit numbers can we make using only the digits 1, 2, and 3? Answer: 27. Here they are, listed very systematically. (If possible, project this table onto a screen where everyone can see it and look at it for a moment, to see the pattern and how it is generated.)

[4] This content is available online at <http://cnx.org/content/m19462/1.2/>.

First Digit	**Second Digit**	**Third Digit**	**Resulting Number**
		1	111
	1	2	112
		3	113
		1	121
1	2	2	122
		3	123
		1	131
	3	2	132
		3	133
		1	211
	1	2	212
		3	213
		1	221
2	2	2	222
		3	223
		1	231
	3	2	232
		3	233
		1	311
	1	2	312
		3	313
		1	321
3	2	2	322
		3	323
		1	331
	3		

continued on next page

		2	332
		3	333

Table 14.1

Effective, and not particularly difficult...but tedious. How could we have answered without the table? Well, of course, it's the rule of multiplication again. There 3 possibilities for the first digit. For each of these, there are 3 possibilities for the second digit; and for each of these, 3 possibilities for the third digit. $3 \times 3 \times 3 = 27$.

Now, let's ask a different problem: how many possible 3-digit numbers can be made using the digits 1, 2, and 3, if every digit is used only once? Once again, we can list them systematically—and it's a lot easier this time. Once you have chosen the first two digits, the third digit is forced. There are only six possibilities.

First Digit	**Second Digit**	**Third Digit**	**Number**
1	2	3	123
	3	2	132
2	1	3	213
	3	1	231
3	1	2	312
	2	1	321

Table 14.2

I really do believe it is important to show them these tables before doing any calculations!!! There is no substitute for seeing everything laid out in an organized manner to get a feeling for the space.

Once again, however, once they have seen the table, we can ask the question: why 6? And once again, we can answer that question using the rule of multiplication. There are three possible numbers that can go in the first digit. Once you have chosen that digit, there are only two possible numbers that can go in the second digit. And once you have chosen that, there is only one number that can possibly go in the third. $3 \times 2 \times 1 = 6$.

Exercise 14.4.1

Repeat the above problems, only with nine digits instead of three. First, how many different nine-digit numbers can be made using the digits 1–9? Second, how many different nine-digit numbers can be made if you use the digits 1–9, but use each digit only once? Obviously we don't want to make these tables (even the second one is prohibitive!) but with the rule of multiplication, and our calculators, we can figure out how big the tables would be. Give them a couple of minutes on this.

The first is $9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9$. (Nine possibilities for the first digit; for each of those, nine for the second; and so on.) Even that is tedious to write. Let's write it like this instead: 99. We can punch it into the calculator just like that.

The second is $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$. (Nine possibilities for the first digit; for each of those, only eight for the second, because one of them is used up; and so on.) Is there any easy way to write that? In fact, there is. It is called 9 factorial, and it is written 9! You may want to show them how to get factorials on their calculators. On some (such as the TI-83) the factorial option is actually listed under probability, reflecting the fact that factorials are used so often in probability problems, for this very reason.

Incidentally, $9^9 = 387,420,489$ possibilities for the first scenario. $9! = 362,880$ for the second: still a pretty big number, but only about a thousandth as big as the first one. This should come as no surprise: almost all of the nine-digit numbers use the same digit twice somewhere or other!

Exercise 14.4.2

Question: How many different ways can five books be arranged on a shelf? Give them a minute, and see if they can figure out that it is the same problem we just did. 5 books can go in the first position; for each of these, 4 in the second position; and so on. $5! = 120$ possibilities.

Exercise 14.4.3

Question: How many three-digit numbers can be made using the digits 1–9?

If we are allowed to repeat digits, this is hopefully pretty easy by this point: $9 \times 9 \times 9$. Written more concisely, 9^3.

But what if we're not? Is there any way we can write $9 \times 8 \times 7$ more concisely? There is, and it's a bit sneaky: it is $\frac{9!}{6!}$. Explain why this works. Point out that, while they may not particularly need it for $9 \times 8 \times 7$, it's really nice as a shortcut for $20 \times 19 \times 18...8$.

If you have extra time, ask everyone in class to come up with two scenarios: one of the "the same thing can be used twice" (exponential) variety, and one of the "the same thing cannot be used twice" (factorial) variety.

Homework

"Homework: Permutations"

14.5 Combinations[5]

Once again, this one is lecture, without an in-class worksheet, walking through a series of questions.

Suppose you have a Daisy, an Iris, a Lily, a Rose, and a Violet. You are going to make a floral arrangement with three of them. How many possible arrangements can you make?

Based on yesterday, it's tempting to answer $5 \times 4 \times 3$, or $\frac{5!}{2!}$. But here's why this is different from yesterday's problems: order doesn't matter. "A rose, a daisy, and an iris" is the same arrangement as "A daisy, an iris, and a rose": you don't want to count it twice.

So, let's start the way we did yesterday: list them all. Give the class a minute to do this. Remind them, as always, that it's best to be systematic to make sure you list every possibility exactly once. Here's my list.

First Flower	**Second Flower**	**Third Flower**	**Arrangement**
Daisy	Iris	Lily	Daisy, Iris, Lily
		Rose	Daisy, Iris, Rose
		Violet	Daisy, Iris, Violet
	Lily	Rose	Daisy, Lily, Rose
		Violet	Daisy, Lily, Violet
	Rose	Violet	Daisy, Rose, Violet

continued on next page

[5]This content is available online at <http://cnx.org/content/m19460/1.1/>.

Iris	Lily	Rose	Iris, Lily, Rose
		Violet	Iris, Lily, Violet
	Rose	Violet	Iris, Rose, Violet
Lily	Rose	Violet	Lily, Rose, Violet

Table 14.3

10 items in all. I generated the list by using the same kind of systematic approach I used for the permutations, but always moving forward in the list: so after "Lily" I'm allowed to list "Rose" and "Violet," but not "Daisy" or "Iris."

This turns out to be such a common and important operation that it gets its own name: "choose." We would say "5 choose 3 is 10," sometimes written $\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10$. It means, if you have five items, and want to choose 3 of them, there are ten ways to do so.

Exercise 14.5.1

The four Beatles are John, Paul, Georg, and Ringo. Suppose you are going to put photographs of three of them on your door. List all the possible combinations. How many are there?

Give the class a minute to list—they should make a diagram like the one I made above!—and count. When they are done, you can point out that the answer (four combinations) is very obvious if you look at it backward: each combination leaves exactly one Beatle out. There is a very important insight here, which can also be applied to the flower example: each arrangement listed leaves exactly two flowers out. So 5 choose 3 ("which flowers should I include?") is the same number as 5 choose 2 ("which flowers should I leave out?").

Now we're going to try something with bigger numbers. A drama teacher looks out at a class of 30 students, and wants to choose 2 of them to run a scene. How many possible pairs of students are there? As before, when the numbers get this large, we can't reasonably list all the combinations: we need an algebraic way of figuring out how many there are, without actually counting them all.

To start off, let's turn this combinations ("order doesn't matter") problem into a permutations ("order does matter") problem. The teacher wants to choose one student to play the Child and one to play the Dog. In this version of the problem, "John plays the Child and Susan plays the Dog" is different from "Susan plays the Child and John plays the Dog," and should be counted separately. So it is a straightforward permutations problem. There are 30 possible actors for the Child, and for each of those, 29 for the Dog. $30 \times 29 = 870$ possible scenes.

Now let's return to the original problem, how many pairs of students are there? In this problem, "John-Susan" and "Susan-John" are the same pair. The key insight here is that, when we ran the permutations problem, we counted each pair twice. So the answer is $870/2 = 435$ pairs.

Take this slow and easy. This is a very general approach to combinations problem. First, you solve the (easier) permutations problem. Then you ask, "How many times did I count every group?" and divide by that.

Ask the class to try this approach on the original (flower) problem. They should answer three questions.

1. How many permutations are there? Remember that in this question, "rose–daisy–iris" and "daisy–iris–rose" are two different arrangements, as if each flower is being placed in a numbered slot.
2. Now, when we counted up the permutations, how many times did they redundantly count each combination?
3. Divide the first answer by the second, and you have the total number of combinations.

Lay out the process, then have them work the problem. Many of them will get stuck on step (2), incorrectly thinking that we counted each permutation three times. In fact, we counted each one six times:

daisy–iris–rose, daisy–rose–iris, iris–daisy–rose, iris–rose–daisy, rose–daisy–iris, rose–iris–daisy

The permutations are $5 \times 4 \times 3$, or $\frac{5!}{2!}$, or 60. But each combination is listed six times, so the total number of combinations is 10, as we counted before.

So...why six times? This is the last question, and it's a hard one. When we were choosing two items, we counted each combination twice (John–Susan, Susan–John). When we were choosing three items, we counted each combination six times. What's the pattern? Give them a minute to think about this. Then make sure they understand that it is, in fact...another permutations problem! The list of six I gave above is simply the number of ways you can arrange three items, or (3!). You would divide by (4!) for 4 items, and so on.

Homework

"Homework: Permutations and Combinations"

A sample test, and real test, and you're done!

Solutions to Exercises in Chapter 14

Solution to Exercise 14.1.4 (p. 110)

It's six (number of possibilities for the die) times two (number of possibilities for the coin). For the second question on the worksheet, with the frogs, there are 15,000 groups, or 5,000×3. This multiplication rule is really the heart of all probability work, so it's best to get used to it early.

Index of Keywords and Terms

Keywords are listed by the section with that keyword (page numbers are in parentheses). Keywords do not necessarily appear in the text of the page. They are merely associated with that section. *Ex.* apples, § 1.1 (1) **Terms** are referenced by the page they appear on. *Ex.* apples, 1

Attributions

Collection: *Advanced Algebra II: Teacher's Guide*
Edited by: Kenny M. Felder
URL: http://cnx.org/content/col10687/1.3/
License: http://creativecommons.org/licenses/by/2.0/

Module: "The Long Rambling Philosophical Introduction (*but please read it anyway)"
Used here as: "The Long Rambling Philosophical Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19503/1.1/
Pages: 1-3
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "How to use Advanced Algebra II"
Used here as: "How to Use Advanced Algebra II"
By: Kenny M. Felder
URL: http://cnx.org/content/m19435/1.6/
Pages: 5-19
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Functions Guide – Introduction"
Used here as: "Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19335/1.2/
Page: 25
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Functions Guide – The Function Game"
Used here as: "The Function Game"
By: Kenny M. Felder
URL: http://cnx.org/content/m19342/1.2/
Pages: 25-26
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Functions Guide – "The Real World""
Used here as: ""The Real World""
By: Kenny M. Felder
URL: http://cnx.org/content/m19331/1.2/
Pages: 26-27
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Functions Guide – Algebraic Generalizations"
Used here as: "Algebraic Generalizations"
By: Kenny M. Felder
URL: http://cnx.org/content/m19332/1.3/
Pages: 27-28
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Functions Guide – Graphing"
Used here as: "Graphing"
By: Kenny M. Felder
URL: http://cnx.org/content/m19334/1.3/
Pages: 28-30
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Functions Guide – Permutations"
Used here as: "Permutations"
By: Kenny M. Felder
URL: http://cnx.org/content/m19339/1.3/
Page: 30
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Functions Guide – Test Preparation"
Used here as: "Test Preparation"
By: Kenny M. Felder
URL: http://cnx.org/content/m19340/1.2/
Pages: 31-32
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Functions Guide – Lines"
Used here as: "Lines"
By: Kenny M. Felder
URL: http://cnx.org/content/m19337/1.2/
Page: 32
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Functions Guide – Composite Functions"
Used here as: "Composite Functions"
By: Kenny M. Felder
URL: http://cnx.org/content/m19333/1.2/
Pages: 32-33
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Functions Guide – Inverse Functions"
Used here as: "Inverse Functions"
By: Kenny M. Felder
URL: http://cnx.org/content/m19336/1.2/
Pages: 33-35
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Inequalities and Absolute Values – Introduction"
Used here as: "Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19432/1.3/
Page: 37
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Inequalities and Absolute Values – Inequalities"
Used here as: "Inequalities"
By: Kenny M. Felder
URL: http://cnx.org/content/m19430/1.2/
Page: 37
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Inequalities and Absolute Values – Inequality Word Problems"
Used here as: "Inequality Word Problems"
By: Kenny M. Felder
URL: http://cnx.org/content/m19428/1.2/
Page: 38
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Inequalities and Absolute Values – Absolute Value Equations"
Used here as: "Absolute Value Equations"
By: Kenny M. Felder
URL: http://cnx.org/content/m19426/1.2/
Page: 38
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Inequalities and Absolute Values – Absolute Value Inequalities"
Used here as: "Absolute Value Inequalities"
By: Kenny M. Felder
URL: http://cnx.org/content/m19431/1.2/
Pages: 38-39
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Inequalities and Absolute Values – Graphing Inequalities and Absolute Values"
Used here as: "Graphing Inequalities and Absolute Values"
By: Kenny M. Felder
URL: http://cnx.org/content/m19433/1.2/
Pages: 39-40
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Simultaneous Equations – Introduction to Simultaneous Equations"
Used here as: "Introduction to Simultaneous Equations"
By: Kenny M. Felder
URL: http://cnx.org/content/m19497/1.2/
Page: 41
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Simultaneous Equations – Simultaneous Equations"
Used here as: "Simultaneous Equations"
By: Kenny M. Felder
URL: http://cnx.org/content/m19498/1.2/
Page: 41
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Simultaneous Equations – The "Generic" Simultaneous Equation"
Used here as: "The "Generic" Simultaneous Equation"
By: Kenny M. Felder
URL: http://cnx.org/content/m19499/1.2/
Page: 42
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Quadratic Equations – Introduction"
Used here as: "Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19469/1.2/
Page: 43
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Quadratic Equations – Multiplying Binomials"
Used here as: "Multiplying Binomials"
By: Kenny M. Felder
URL: http://cnx.org/content/m19472/1.2/
Pages: 43-44
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Quadratic Equations – Factoring"
Used here as: "Factoring"
By: Kenny M. Felder
URL: http://cnx.org/content/m19466/1.2/
Page: 44
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Quadratic Equations – Introduction to Quadratic Equations"
Used here as: "Introduction to Quadratic Equations"
By: Kenny M. Felder
URL: http://cnx.org/content/m19470/1.2/
Pages: 44-45
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Quadratic Equations – Completing the Square"
Used here as: "Completing the Square"
By: Kenny M. Felder
URL: http://cnx.org/content/m19465/1.2/
Page: 45
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Quadratic Equations – The "Generic" Quadratic Equation"
Used here as: "The "Generic" Quadratic Equation"
By: Kenny M. Felder
URL: http://cnx.org/content/m19480/1.2/
Page: 46
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Quadratic Equations – Graphing Quadratic Functions"
Used here as: "Graphing Quadratic Functions"
By: Kenny M. Felder
URL: http://cnx.org/content/m19468/1.2/
Page: 46
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Quadratic Equations – Graphing Quadratic Functions II"
Used here as: "Graphing Quadratic Functions II"
By: Kenny M. Felder
URL: http://cnx.org/content/m19467/1.2/
Pages: 46-47
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Quadratic Equations – Solving Problems by Graphing Quadratic Functions"
Used here as: "Solving Problems by Graphing Quadratic Functions"
By: Kenny M. Felder
URL: http://cnx.org/content/m19479/1.2/
Page: 47
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Quadratic Equations – Quadratic Inequalities"
Used here as: "Quadratic Inequalities"
By: Kenny M. Felder
URL: http://cnx.org/content/m19473/1.2/
Pages: 47-49
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Exponents – Introduction"
Used here as: "Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19325/1.2/
Page: 51
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Exponents – Rules of Exponents"
Used here as: "Rules of Exponents"
By: Kenny M. Felder
URL: http://cnx.org/content/m19327/1.3/
Page: 51
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Exponents – Extending the Idea of Exponents"
Used here as: "Extending the Idea of Exponents"
By: Kenny M. Felder
URL: http://cnx.org/content/m19318/1.2/
Pages: 51-53
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Exponents – Fractional Exponents"
Used here as: "Fractional Exponents"
By: Kenny M. Felder
URL: http://cnx.org/content/m19322/1.2/
Page: 53
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Exponents – “Real Life” Exponential Curves"
Used here as: "“Real Life” Exponential Curves"
By: Kenny M. Felder
URL: http://cnx.org/content/m19329/1.3/
Page: 54
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Logarithms – Introduction"
Used here as: "Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19436/1.2/
Page: 55
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Logarithms – Properties of Logarithms"
Used here as: "Properties of Logarithms"
By: Kenny M. Felder
URL: http://cnx.org/content/m19438/1.1/
Page: 56
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Logarithms – Using the Laws of Logarithms"
Used here as: "Using the Laws of Logarithms"
By: Kenny M. Felder
URL: http://cnx.org/content/m19440/1.1/
Page: 56
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Logarithms – So What Are Logarithms Good For, Anyway?"
Used here as: "So What Are Logarithms Good For, Anyway?"
By: Kenny M. Felder
URL: http://cnx.org/content/m19439/1.2/
Pages: 56-57
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Rational Expressions – Introduction"
Used here as: "Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19486/1.1/
Page: 59
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Rational Expressions – Rational Expressions"
Used here as: "Rational Expressions"
By: Kenny M. Felder
URL: http://cnx.org/content/m19488/1.1/
Pages: 59-61
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Rational Expressions – Rational Equations"
Used here as: "Rational Equations"
By: Kenny M. Felder
URL: http://cnx.org/content/m19489/1.1/
Page: 61
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Rational Expressions – Polynomial Division"
Used here as: "Polynomial Division"
By: Kenny M. Felder
URL: http://cnx.org/content/m19487/1.2/
Pages: 62-63
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Radicals – Introduction to Radicals"
Used here as: "Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19484/1.2/
Pages: 65-66
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Radicals – A Bunch of Other Stuff About Radicals"
Used here as: "A Bunch of Other Stuff About Radicals"
By: Kenny M. Felder
URL: http://cnx.org/content/m19483/1.1/
Page: 66
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Radicals – Radical Equations"
Used here as: "Radical Equations"
By: Kenny M. Felder
URL: http://cnx.org/content/m19485/1.2/
Pages: 66-67
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Imaginary Numbers – Introduction"
Used here as: "Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19424/1.2/
Pages: 69-70
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Imaginary Numbers – Introduction to Imaginary Numbers"
Used here as: "Introduction to Imaginary Numbers"
By: Kenny M. Felder
URL: http://cnx.org/content/m21990/1.1/
Pages: 70-71
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/3.0/

Module: "Imaginary Numbers – Complex Numbers"
Used here as: "Complex Numbers"
By: Kenny M. Felder
URL: http://cnx.org/content/m19423/1.4/
Pages: 71-72
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Imaginary Numbers – Me, Myself, and the Square Root of i"
Used here as: "Me, Myself, and the Square Root of i"
By: Kenny M. Felder
URL: http://cnx.org/content/m19425/1.2/
Pages: 72-73
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Matrices – Introduction"
Used here as: "Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19445/1.1/
Pages: 75-76
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Matrices – Multiplying Matrices I (row × column)"
Used here as: "Multiplying Matrices I (row × column)"
By: Kenny M. Felder
URL: http://cnx.org/content/m19448/1.1/
Page: 76
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Matrices – Multiplying Matrices II (the full monty)"
Used here as: "Multiplying Matrices II (the full monty)"
By: Kenny M. Felder
URL: http://cnx.org/content/m19449/1.2/
Pages: 76-77
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Matrices – Identity and Inverse Matrices"
Used here as: "Identity and Inverse Matrices"
By: Kenny M. Felder
URL: http://cnx.org/content/m19443/1.1/
Page: 77
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Matrices – Inverse of the Generic 2x2 Matrix"
Used here as: "Inverse of the Generic 2x2 Matrix"
By: Kenny M. Felder
URL: http://cnx.org/content/m19446/1.1/
Pages: 77-78
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Matrices – Use Matrices for Transformation"
Used here as: "Use Matrices for Transformation"
By: Kenny M. Felder
URL: http://cnx.org/content/m19451/1.1/
Page: 78
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Matrices – Matrices on the Calculator"
Used here as: "Matrices on the Calculator"
By: Kenny M. Felder
URL: http://cnx.org/content/m19447/1.1/
Page: 78
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Matrices – Determinants"
Used here as: "Determinants"
By: Kenny M. Felder
URL: http://cnx.org/content/m19442/1.1/
Page: 79
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Matrices – Solving Linear Equations"
Used here as: "Solving Linear Equations"
By: Kenny M. Felder
URL: http://cnx.org/content/m19450/1.1/
Pages: 79-80
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Modeling Data with Functions – Introduction"
Used here as: "Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19454/1.2/
Page: 81
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Modeling Data with Functions – Direct and Inverse Variation"
Used here as: "Direct and Inverse Variation"
By: Kenny M. Felder
URL: http://cnx.org/content/m19452/1.1/
Pages: 81-83
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Modeling Data with Functions – Calculator Regression"
Used here as: "Calculator Regression"
By: Kenny M. Felder
URL: http://cnx.org/content/m19453/1.1/
Pages: 83-84
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Conic Sections Guide – Introduction"
Used here as: "Introduction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19307/1.1/
Page: 85
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Conic Sections Guide – Distance"
Used here as: "Distance"
By: Kenny M. Felder
URL: http://cnx.org/content/m19299/1.2/
Pages: 85-86
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Conic Sections Guide – Circles"
Used here as: "Circles"
By: Kenny M. Felder
URL: http://cnx.org/content/m19298/1.2/
Pages: 86-87
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Conic Sections Guide – Parabolas, Day 1"
Used here as: "Parabolas, Day 1"
By: Kenny M. Felder
URL: http://cnx.org/content/m19315/1.2/
Page: 87
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Conic Sections Guide – Parabolas, Day 2"
Used here as: "Parabolas, Day 2"
By: Kenny M. Felder
URL: http://cnx.org/content/m19313/1.2/
Pages: 88-90
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Conic Sections Guide – Parabolas: From Definition to Equation"
Used here as: "Parabolas: From Definition to Equation"
By: Kenny M. Felder
URL: http://cnx.org/content/m19311/1.2/
Pages: 90-91
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Conic Sections Guide – Ellipses"
Used here as: "Ellipses"
By: Kenny M. Felder
URL: http://cnx.org/content/m19303/1.2/
Pages: 91-94
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Conic Sections Guide – Ellipses: From Definition to Equation"
Used here as: "Ellipses: From Definition to Equation"
By: Kenny M. Felder
URL: http://cnx.org/content/m19305/1.1/
Pages: 95-97
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Conic Sections Guide – Hyperbolas"
Used here as: "Hyperbolas"
By: Kenny M. Felder
URL: http://cnx.org/content/m19306/1.2/
Pages: 97-100
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Sequences and Series Guide – Prerequisites"
Used here as: "Prerequisites"
By: Kenny M. Felder
URL: http://cnx.org/content/m19495/1.1/
Page: 101
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Sequences and Series Guide – Arithmetic and Geometric Sequences"
Used here as: "Arithmetic and Geometric Sequences"
By: Kenny M. Felder
URL: http://cnx.org/content/m19490/1.1/
Pages: 101-102
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Sequences and Series Guide – Series and Series Notation"
Used here as: "Series and Series Notation"
By: Kenny M. Felder
URL: http://cnx.org/content/m19491/1.1/
Page: 102
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Sequences and Series Guide – Arithmetic and Geometric Series"
Used here as: "Arithmetic and Geometric Series"
By: Kenny M. Felder
URL: http://cnx.org/content/m19494/1.1/
Pages: 102-105
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Sequences and Series Guide – Proof by Induction"
Used here as: "Proof by Induction"
By: Kenny M. Felder
URL: http://cnx.org/content/m19492/1.1/
Pages: 105-106
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Sequences and Series Guide – Extra Credit"
Used here as: "Extra Credit"
By: Kenny M. Felder
URL: http://cnx.org/content/m19493/1.1/
Pages: 106-107
Copyright: Kenny M. Felder
License: http://creativecommons.org/licenses/by/2.0/

Module: "Probability – Tree Diagrams"
Used here as: "Tree Diagrams"
By: Kenny M. Felder
URL: http://cnx.org/content/m19463/1.1/
Pages: 109-111

Module: "Probability – Introduction to Probability"
Used here as: "Introduction to Probability"
By: Kenny M. Felder
URL: http://cnx.org/content/m19461/1.1/
Pages: 111-112

Module: "Probability – Trickier Probability Problems"
Used here as: "Trickier Probability Problems"
By: Kenny M. Felder
URL: http://cnx.org/content/m19464/1.2/
Pages: 112-113

Module: "Probability – Permutations"
Used here as: "Permutations"
By: Kenny M. Felder
URL: http://cnx.org/content/m19462/1.2/
Pages: 113-116

Module: "Probability – Combinations"
Used here as: "Combinations"
By: Kenny M. Felder
URL: http://cnx.org/content/m19460/1.1/
Pages: 116-118

Advanced Algebra II: Teacher's Guide
This is the Teacher's Guide for Kenny Felder's course in Advanced Algebra II. This guide is *not* an answer key for the homework problems: rather, it is a day-by-day guide to help the teacher understand how the author envisions the materials being used. This text is designed for use with the "Advanced Algebra II: Conceptual Explanations" (http://cnx.org/content/col10624/latest/) and the "Advanced Algebra II: Homework and Activities" (http://cnx.org/content/col10686/latest/) collections to make up the entire course.