Classic Fourier Series*

Don Johnson

Abstract

Signals can be composed by a superposition of an infinite number of sine and cosine functions. The coefficients of the superposition depend on the signal being represented and are equivalent to knowing the function itself.

The classic Fourier series as derived originally expressed a periodic signal (period T) in terms of harmonically related sines and cosines.

$$s(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi kt}{T}\right) \tag{1}$$

The complex Fourier series and the sine-cosine series are identical, each representing a signal's spectrum. The **Fourier coefficients**, a_k and b_k, express the real and imaginary parts respectively of the spectrum while the coefficients c_k of the complex Fourier series express the spectrum as a magnitude and phase. Equating the classic Fourier series (1) to the complex Fourier series, an extra factor of two and complex conjugate become necessary to relate the Fourier coefficients in each.

$$c_k = \frac{1}{2}(a_k - ib_k)$$

Exercise 1 ***(Solution on p. 4.)***

Derive this relationship between the coefficients of the two Fourier series.

Just as with the complex Fourier series, we can find the Fourier coefficients using the **orthogonality** properties of sinusoids. Note that the cosine and sine of harmonically related frequencies, even the **same** frequency, are orthogonal.

$$\forall k, l, k \in \mathbb{Z} l \in \mathbb{Z} : \left(\int_0^T \sin\left(\frac{2\pi kt}{T}\right)\cos\left(\frac{2\pi lt}{T}\right)dt = 0\right) \tag{2}$$

$$\int_0^T \sin\left(\frac{2\pi kt}{T}\right)\sin\left(\frac{2\pi lt}{T}\right)dt = \begin{cases} \frac{T}{2} \text{ if } (k = l) \wedge (k \neq 0) \wedge (l \neq 0) \\ 0 \text{ if } (k \neq l) \vee (k = 0 = l) \end{cases}$$

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$$\int_0^T \cos\left(\frac{2\pi kt}{T}\right)\cos\left(\frac{2\pi lt}{T}\right)dt = \begin{cases} \frac{T}{2} \text{ if } (k=l) \wedge (k\neq 0) \wedge (l\neq 0) \\ T \text{ if } k=0=l \\ 0 \text{ if } k\neq l \end{cases}$$

These orthogonality relations follow from the following important trigonometric identities.

$$\begin{aligned} \sin(\alpha)\sin(\beta) &= \tfrac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)) \\ \cos(\alpha)\cos(\beta) &= \tfrac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)) \\ \sin(\alpha)\cos(\beta) &= \tfrac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta)) \end{aligned} \tag{3}$$

These identities allow you to substitute a sum of sines and/or cosines for a product of them. Each term in the sum can be integrating by noticing one of two important properties of sinusoids.

- The integral of a sinusoid over an **integer** number of periods equals zero.
- The integral of the **square** of a unit-amplitude sinusoid over a period T equals $\frac{T}{2}$.

To use these, let's, for example, multiply the Fourier series for a signal by the cosine of the l^{th} harmonic $\cos\left(\frac{2\pi lt}{T}\right)$ and integrate. The idea is that, because integration is linear, the integration will sift out all but the term involving a_l.

$$\int_0^T s(t)\cos\left(\frac{2\pi lt}{T}\right)dt = \int_0^T a_0\cos\left(\frac{2\pi lt}{T}\right)dt + \sum_{k=1}^{\infty} a_k \int_0^T \cos\left(\frac{2\pi kt}{T}\right)\cos\left(\frac{2\pi lt}{T}\right)dt + \sum_{k=1}^{\infty} b_k \int_0^T \sin\left(\frac{2\pi kt}{T}\right)\cos\left(\frac{2\pi lt}{T}\right)dt \tag{4}$$

The first and third terms are zero; in the second, the only non-zero term in the sum results when the indices k and l are equal (but not zero), in which case we obtain $\frac{a_l T}{2}$. If $k=0=l$, we obtain $a_0 T$. Consequently,

$$\forall l, l\neq 0 : \left(a_l = \frac{2}{T}\int_0^T s(t)\cos\left(\frac{2\pi lt}{T}\right)dt\right)$$

All of the Fourier coefficients can be found similarly.

$$\begin{aligned} & a_0 = \tfrac{1}{T}\int_0^T s(t)\,dt \\ & \forall k, k\neq 0 : \left(a_k = \tfrac{2}{T}\int_0^T s(t)\cos\left(\tfrac{2\pi kt}{T}\right)dt\right) \\ & b_k = \tfrac{2}{T}\int_0^T s(t)\sin\left(\tfrac{2\pi kt}{T}\right)dt \end{aligned} \tag{5}$$

Exercise 2 *(Solution on p. 4.)*

The expression for a_0 is referred to as the **average value** of $s(t)$. Why?

Exercise 3 *(Solution on p. 4.)*

What is the Fourier series for a unit-amplitude square wave?

Example 1

Let's find the Fourier series representation for the half-wave rectified sinusoid.

$$s(t) = \begin{cases} \sin\left(\frac{2\pi t}{T}\right) \text{ if } 0\leq t<\frac{T}{2} \\ 0 \text{ if } \frac{T}{2}\leq t<T \end{cases} \tag{6}$$

Begin with the sine terms in the series; to find b_k we must calculate the integral

$$b_k = \frac{2}{T}\int_0^{\frac{T}{2}} \sin\left(\frac{2\pi t}{T}\right)\sin\left(\frac{2\pi kt}{T}\right)dt \tag{7}$$

Using our trigonometric identities turns our integral of a product of sinusoids into a sum of integrals of individual sinusoids, which are much easier to evaluate.

$$\begin{aligned}\int_0^{\frac{T}{2}} \sin\left(\frac{2\pi t}{T}\right)\sin\left(\frac{2\pi k t}{T}\right)dt &= \frac{1}{2}\int_0^{\frac{T}{2}} \cos\left(\frac{2\pi(k-1)t}{T}\right) - \cos\left(\frac{2\pi(k+1)t}{T}\right)dt \\ &= \begin{cases} \frac{1}{2} & \text{if } k=1 \\ 0 & \text{otherwise}\end{cases}\end{aligned} \quad (8)$$

Thus,

$$b_1 = \frac{1}{2}$$

$$b_2 = b_3 = \dots = 0$$

On to the cosine terms. The average value, which corresponds to a_0, equals $\frac{1}{\pi}$. The remainder of the cosine coefficients are easy to find, but yield the complicated result

$$a_k = \begin{cases} -\left(\frac{2}{\pi}\frac{1}{k^2-1}\right) & \text{if } k \in \{2,4,\dots\} \\ 0 & \text{if k odd}\end{cases} \quad (9)$$

Thus, the Fourier series for the half-wave rectified sinusoid has non-zero terms for the average, the fundamental, and the even harmonics.

Solutions to Exercises in this Module

Solution to Exercise (p. 1)
Write the coefficients of the complex Fourier series in Cartesian form as $c_k = A_k + iB_k$ and substitute into the expression for the complex Fourier series.

$$\sum_{k=-\infty}^{\infty} c_k e^{i\frac{2\pi kt}{T}} = \sum_{k=-\infty}^{\infty} (A_k + iB_k)\, e^{i\frac{2\pi kt}{T}}$$

Simplifying each term in the sum using Euler's formula,

$$\begin{aligned}(A_k + iB_k)\, e^{i\frac{2\pi kt}{T}} &= (A_k + iB_k)\left(\cos\left(\frac{2\pi kt}{T}\right) + i\sin\left(\frac{2\pi kt}{T}\right)\right)\\ &= A_k\cos\left(\frac{2\pi kt}{T}\right) - B_k\sin\left(\frac{2\pi kt}{T}\right) + i\left(A_k\sin\left(\frac{2\pi kt}{T}\right) + B_k\cos\left(\frac{2\pi kt}{T}\right)\right)\end{aligned}$$

We now combine terms that have the same frequency index **in magnitude**. Because the signal is real-valued, the coefficients of the complex Fourier series have conjugate symmetry: $c_{-k} = \overline{c_k}$ or $A_{-k} = A_k$ and $B_{-k} = -B_k$. After we add the positive-indexed and negative-indexed terms, each term in the Fourier series becomes $2A_k\cos\left(\frac{2\pi kt}{T}\right) - 2B_k\sin\left(\frac{2\pi kt}{T}\right)$. To obtain the classic Fourier series (1), we must have $2A_k = a_k$ and $2B_k = -b_k$.

Solution to Exercise (p. 2)
The average of a set of numbers is the sum divided by the number of terms. Viewing signal integration as the limit of a Riemann sum, the integral corresponds to the average.

Solution to Exercise (p. 2)
We found that the complex Fourier series coefficients are given by $c_k = \frac{2}{i\pi k}$. The coefficients are pure imaginary, which means $a_k = 0$. The coefficients of the sine terms are given by $b_k = -(2\Im(c_k))$ so that

$$b_k = \begin{cases} \frac{4}{\pi k} & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even}\end{cases}$$

Thus, the Fourier series for the square wave is

$$\mathrm{sq}(t) = \sum_{k\in\{1,3,\dots\}} \frac{4}{\pi k}\sin\left(\frac{2\pi kt}{T}\right) \quad (10)$$