Discrete-Time Fourier Transform Pair*

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Abstract

Computing discrete-time frequencies by the use of Fourier transforms.

When we obtain the discrete-time signal via sampling an analog signal, the Nyquist frequency corresponds to the discrete-time frequency $\frac{1}{2}$. To show this, note that a sinusoid at the Nyquist frequency $\frac{1}{2T_s}$ has a sampled waveform that equals

Sinusoid at Nyquist Frequency 1/2T

$$\begin{aligned} \cos\left(2\pi\frac{1}{2T_s}nT_s\right) &= \cos(\pi n) \\ &= (-1)^n \end{aligned} \quad (1)$$

The exponential in the DTFT at frequency $\frac{1}{2}$ equals $e^{\frac{-(i2\pi n)}{2}} = e^{-(i\pi n)} = (-1)^n$, meaning that the correspondence between analog and discrete-time frequency is established:

Analog, Discrete-Time Frequency Relationship

$$f_D = f_A T_s \quad (2)$$

where f_D and f_A represent discrete-time and analog frequency variables, respectively. The aliasing figure[1] provides another way of deriving this result. As the duration of each pulse in the periodic sampling signal $p_{T_s}(t)$ narrows, the amplitudes of the signal's spectral repetitions, which are governed by the Fourier series coefficients of $p_{T_s}(t)$, become increasingly equal. [2] Thus, the sampled signal's spectrum becomes periodic with period $\frac{1}{T_s}$. Thus, the Nyquist frequency $\frac{1}{2T_s}$ corresponds to the frequency $\frac{1}{2}$.

The inverse discrete-time Fourier transform is easily derived from the following relationship:

$$\int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} e^{-(i2\pi fm)} e^{+i\pi fn} df = \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases} \quad (3)$$

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[1] "The Sampling Theorem", Figure 2: aliasing <http://cnx.org/content/m0050/latest/#alias>

[2] Examination of the periodic pulse signal reveals that as Δ decreases, the value of c_0 , the largest Fourier coefficient, decreases to zero: $|c_0| = \frac{A\Delta}{T}$. Thus, to maintain a mathematically viable Sampling Theorem, the amplitude A must increase as $\frac{1}{\Delta}$, becoming infinitely large as the pulse duration decreases. Practical systems use a small value of Δ , say $0.1T_s$ and use amplifiers to rescale the signal.

Therefore, we find that

$$\begin{aligned} \int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} S\left(e^{i2\pi f}\right) e^{+i2\pi f n} df &= \int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} \sum_m \left(s\left(m\right) e^{-\left(i2\pi f m\right)} e^{+i2\pi f n}\right) df \\ &= \sum_m \left(s\left(m\right) \int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} e^{\left(-\left(i2\pi f\right)\right)\left(m-n\right)} df\right) \\ &= s\left(n\right) \end{aligned} \quad (4)$$

The Fourier transform pairs in discrete-time are

Fourier Transform Pairs in Discrete Time

$$S\left(e^{i2\pi f}\right) = \sum_n \left(s\left(n\right) e^{-\left(i2\pi f n\right)}\right) \quad (5)$$

Fourier Transform Pairs in Discrete Time

$$s\left(n\right) = \int_{-\left(\frac{1}{2}\right)}^{\frac{1}{2}} S\left(e^{i2\pi f}\right) e^{+i2\pi f n} df \quad (6)$$