The Complex Exponential*

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Abstract

Describes the complex exponential function.

1 The Exponential Basics

The **complex exponential** is one of the most fundamental and important signal in signal and system analysis. Its importance comes from its functions as a basis for periodic signals as well as being able to characterize linear, time-invariant[1] signals. Before proceeding, you should be familiar with the ideas and functions of complex numbers[2].

1.1 Basic Exponential

For all numbers x, we easily derive and define the **exponential function** from the Taylor's series below:

$$e^{x}=1+\frac{x^{1}}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\dots \tag{1}$$

$$e^{x}=\sum_{k=0}^{\infty}\left(\frac{1}{k!}x^{k}\right) \tag{2}$$

We can prove, using the ratio test, that this series does indeed converge. Therefore, we can state that the exponential function shown above is continuous and easily defined.

From this definition, we can prove the following property for exponentials that will be very useful, especially for the complex exponentials discussed in the next section.

$$e^{x_{1}+x_{2}}=\left(e^{x_{1}}\right)\left(e^{x_{2}}\right) \tag{3}$$

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[1]"System Classifications and Properties" <http://cnx.org/content/m10084/latest/>
[2]"Complex Numbers" <http://cnx.org/content/m0081/latest/>

1.2 Complex Continuous-Time Exponential

Now for all complex numbers s, we can define the **complex continuous-time exponential signal** as

$$\begin{aligned} f(t) &= Ae^{st} \\ &= Ae^{i\omega t} \end{aligned} \quad (4)$$

where A is a constant, t is our independent variable for time, and for s imaginary, $s = i\omega$. Finally, from this equation we can reveal the ever important **Euler's Identity** (for more information on Euler read this short biography[3]):

$$Ae^{i\omega t} = A\cos(\omega t) + i\,(A\sin(\omega t)) \quad (5)$$

From Euler's Identity we can easily break the signal down into its real and imaginary components. Also we can see how exponentials can be combined to represent any real signal. By modifying their frequency and phase, we can represent any signal through a superposity of many signals - all capable of being represented by an exponential.

The above expressions do not include any information on phase however. We can further generalize our above expressions for the exponential to generalize sinusoids with any phase by making a final substitution for s, $s = \sigma + i\omega$, which leads us to

$$\begin{aligned} f(t) &= Ae^{st} \\ &= Ae^{(\sigma + i\omega)t} \\ &= Ae^{\sigma t}e^{i\omega t} \end{aligned} \quad (6)$$

where we define S as the **complex amplitude**, or **phasor**, from the first two terms of the above equation as

$$S = Ae^{\sigma t} \quad (7)$$

Going back to Euler's Identity, we can rewrite the exponentials as sinusoids, where the phase term becomes much more apparent.

$$f(t) = Ae^{\sigma t}(\cos(\omega t) + i\sin(\omega t)) \quad (8)$$

As stated above we can easily break this formula into its real and imaginary part as follows:

$$\Re(f(t)) = Ae^{\sigma t}\cos(\omega t) \quad (9)$$

$$\Im(f(t)) = Ae^{\sigma t}\sin(\omega t) \quad (10)$$

1.3 Complex Discrete-Time Exponential

Finally we have reached the last form of the exponential signal that we will be interested in, the **discrete-time exponential signal**, which we will not give as much detail about as we did for its continuous-time counterpart, because they both follow the same properties and logic discussed above. Because it is discrete, there is only a slightly different notation used to represents its discrete nature

$$\begin{aligned} f[n] &= Be^{snT} \\ &= Be^{i\omega nT} \end{aligned} \quad (11)$$

where nT represents the discrete-time instants of our signal.

[3] http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Euler.html

2 Euler's Relation

Along with Euler's Identity, Euler also described a way to represent a complex exponential signal in terms of its real and imaginary parts through **Euler's Relation**:

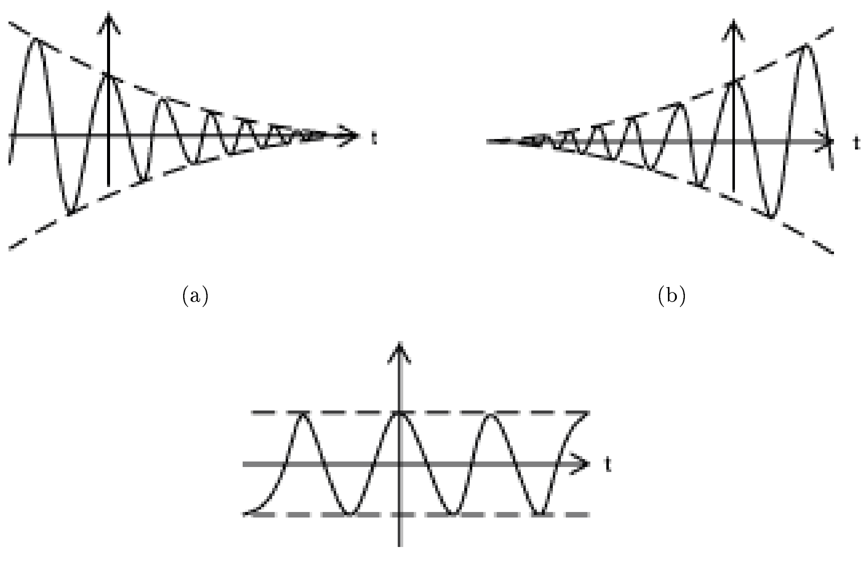
$$\cos(\omega t) = \frac{e^{iwt} + e^{-(iwt)}}{2} \tag{12}$$

$$\sin(\omega t) = \frac{e^{iwt} - e^{-(iwt)}}{2i} \tag{13}$$

$$e^{iwt} = \cos(\omega t) + i\sin(\omega t) \tag{14}$$

3 Drawing the Complex Exponential

At this point, we have shown how the complex exponential can be broken up into its real part and its imaginary part. It is now worth looking at how we can draw each of these parts. We can see that both the real part and the imaginary part have a sinusoid times a real exponential. We also know that sinusoids oscillate between one and negative one. From this it becomes apparent that the real and imaginary parts of the complex exponential will each oscillate between a window defined by the real exponential part.



(c)

Figure 1: The shapes possible for the real part of a complex exponential. Notice that the oscillations are the result of a cosine, as there is a local maximum at $t = 0$. (a) If σ is negative, we have the case of a decaying exponential window. (b) If σ is positive, we have the case of a growing exponential window. (c) If σ is zero, we have the case of a constant window.

While the σ determines the rate of decay/growth, the ω part determines the rate of the oscillations. This is apparent by noticing that the ω is part of the argument to the sinusoidal part.

Exercise 1 *(Solution on p. 5.)*

What do the imaginary parts of the complex exponentials drawn above look like?

Example 1

The following demonstration allows you to see how the argument changes the shape of the complex exponential. See here[4] for instructions on how to use the demo.

This is an unsupported media type. To view, please see http://cnx.org/content/m10060/2.18/Complex_Exponential.vi

4 The Complex Plane

It becomes extremely useful to view the complex variable s as a point in the complex plane[5] (the **s-plane**).

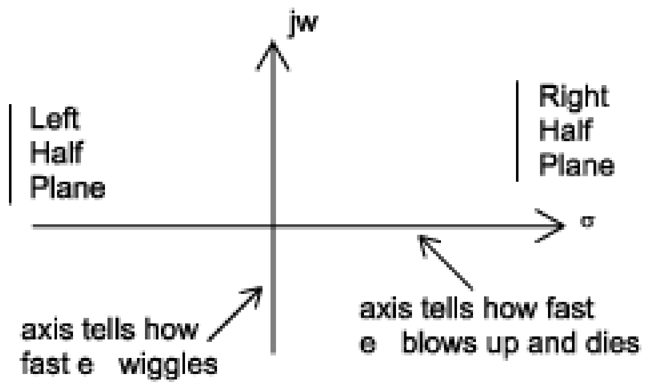


Figure 2: This is the s-plane. Notice that any time s lies in the right half plane, the complex exponential will grow through time, while any time it lies in the left half plane it will decay.

[4] "How to use the LabVIEW demos" <http://cnx.org/content/m11550/latest/>
[5] "The Complex Plane" <http://cnx.org/content/m10596/latest/>

Solutions to Exercises in this Module

Solution to Exercise 1 (p. 4)

They look the same except the oscillation is that of a sinusoid as opposed to a cosinusoid (i.e. it passes through the origin rather than being a local maximum at $t = 0$).