

# CONTINUOUS-TIME CONVOLUTION\*

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## Abstract

Defines convolution and derives the Convolution Integral.

## 1 Motivation

Convolution helps to determine the effect a system has on an input signal. It can be shown that a linear, time-invariant system is completely characterized by its impulse response. At first glance, this may appear to be of little use, since impulse functions are not well defined in real applications. However, the sifting property of impulses tells us that a signal can be decomposed into an infinite sum (integral) of scaled and shifted impulses. By knowing how a system affects a single impulse, and by understanding the way a signal is comprised of scaled and summed impulses, it seems reasonable that it should be possible to scale and sum the impulse responses of a system in order to determine what output signal will result from a particular input. This is precisely what convolution does - **convolution determines the system's output from knowledge of the input and the system's impulse response.**

In the rest of this module, we will examine exactly how convolution is defined from the reasoning above. This will result in the convolution integral (see the next section) and its properties. These concepts are very important in Electrical Engineering and will make any engineer's life a lot easier if the time is spent now to truly understand what is going on.

In order to fully understand convolution, you may find it useful to look at the discrete-time convolution as well. It will also be helpful to experiment with the applets<sup>1</sup> available on the internet. These resources will offer different approaches to this crucial concept.

## 2 Convolution Integral

As mentioned above, the convolution integral provides an easy mathematical way to express the output of an LTI system based on an arbitrary signal,  $x(t)$ , and the system's impulse response,  $h(t)$ . The **convolution integral** is expressed as

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \quad (1)$$

Convolution is such an important tool that it is represented by the symbol  $*$ , and can be written as

$$y(t) = x(t) * h(t) \quad (2)$$

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<sup>1</sup><http://www.jhu.edu/~signals>

By making a simple change of variables into the convolution integral,  $\tau = t - \tau$ , we can easily show that convolution is **commutative**:

$$x(t) * h(t) = h(t) * x(t) \quad (3)$$

For more information on the characteristics of the convolution integral, read about the Properties of Convolution.

We now present two distinct approaches for deriving the convolution integral. These derivations, along with a basic example, will help to build intuition about convolution.

### 3 Derivation I: The Short Approach

The derivation used here closely follows the one discussed in the Motivation (Section 1: Motivation) section above. To begin this, it is necessary to state the assumptions we will be making. In this instance, the only constraints on our system are that it be linear and time-invariant.

#### Brief Overview of Derivation Steps:

1. An impulse input leads to an impulse response output.
2. A shifted impulse input leads to a shifted impulse response output. This is due to the time-invariance of the system.
3. We now scale the impulse input to get a scaled impulse output. This is using the scalar multiplication property of linearity.
4. We can now "sum up" an infinite number of these scaled impulses to get a sum of an infinite number of scaled impulse responses. This is using the additivity attribute of linearity.
5. Now we recognize that this infinite sum is nothing more than an integral, so we convert both sides into integrals.
6. Recognizing that the input is the function  $f(t)$ , we also recognize that the output is exactly the convolution integral.

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**Figure 1:** We begin with a system defined by its impulse response,  $h(t)$ .

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**Figure 2:** We then consider a shifted version of the input impulse. Due to the time invariance of the system, we obtain a shifted version of the output impulse response.

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**Figure 3:** Now we use the scaling part of linearity by scaling the system by a value,  $f(\tau)$ , that is constant with respect to the system variable,  $t$ .

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**Figure 4:** We can now use the additivity aspect of linearity to add an infinite number of these, one for each possible  $\tau$ . Since an infinite sum is exactly an integral, we end up with the integration known as the Convolution Integral. Using the sifting property, we recognize the left-hand side simply as the input  $f(t)$ .

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### 4 Derivation II: The Long Approach

This derivation is really not too different from the one above. It is, however, a little more rigorous and a little longer. Hopefully, if you think you "kind of" get the derivation above, this will help you gain a more complete understanding of convolution.

The first step in this derivation is to define a particular realization of the unit impulse function. For this, we will use  $\delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta} & \text{if } -\frac{\Delta}{2} < t < \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}$

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**Figure 5:** The realization of the unit impulse function that we will use for this derivation.

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After defining our realization of the unit impulse response, we can derive our convolution integral from the following steps found in the table below. Note that the left column represents the input and the right column is the system's output given that input.

### Derivation II of Convolution Integral

Input		Output
$\lim_{\Delta \rightarrow 0} \delta_{\Delta}(t)$	$\rightarrow h \rightarrow$	$\lim_{\Delta \rightarrow 0} h(t)$
$\lim_{\Delta \rightarrow 0} \delta_{\Delta}(t - n\Delta)$	$\rightarrow h \rightarrow$	$\lim_{\Delta \rightarrow 0} h(t - n\Delta)$
$\lim_{\Delta \rightarrow 0} f(n\Delta) \delta_{\Delta}(t - n\Delta) \Delta$	$\rightarrow h \rightarrow$	$\lim_{\Delta \rightarrow 0} f(n\Delta) h(t - n\Delta) \Delta$
$\lim_{\Delta \rightarrow 0} \sum_n f(n\Delta) \delta_{\Delta}(t - n\Delta) \Delta$	$\rightarrow h \rightarrow$	$\lim_{\Delta \rightarrow 0} \sum_n f(n\Delta) h(t - n\Delta) \Delta$
$\int_{-\infty}^{\infty} f(\tau) \delta(t - \tau) d\tau$	$\rightarrow h \rightarrow$	$\int_{-\infty}^{\infty} f(\tau) h(t - \tau) d\tau$
$f(t)$	$\rightarrow h \rightarrow$	$y(t) = \int_{-\infty}^{\infty} f(\tau) h(t - \tau) d\tau$

Table 1

## 5 Implementation of Convolution

Taking a closer look at the convolution integral, we find that we are multiplying the input signal by the time-reversed impulse response and integrating. This will give us the value of the output at one given value of  $t$ . If we then shift the time-reversed impulse response by a small amount, we get the output for another value of  $t$ . Repeating this for every possible value of  $t$ , yields the total output function. While we would never actually do this computation by hand in this fashion, it does provide us with some insight into what is actually happening. We find that we are essentially reversing the impulse response function and sliding it across the input function, integrating as we go. This method, referred to as the **graphical method**, provides us with a much simpler way to solve for the output for simple (contrived) signals, while improving our intuition for the more complex cases where we rely on computers. In fact Texas Instruments<sup>2</sup> develops Digital Signal Processors<sup>3</sup> which have special instruction sets for computations such as convolution.

### Example 1

This demonstration illustrates the graphical method for convolution. See here for instructions on how to use the demo.

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[http://cnx.org/content/m10085/2.25/CT\\_Convolution.llb](http://cnx.org/content/m10085/2.25/CT_Convolution.llb)

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## 6 Basic Example

Let us look at a basic continuous-time convolution example to help express some of the ideas mentioned above through a short example. We will convolve together two unit pulses,  $x(t)$  and  $h(t)$ .

<sup>2</sup><http://www.ti.com>

<sup>3</sup><http://dspvillage.ti.com/docs/toolsoftwarehome.jhtm>



$0 \leq t < 1$ , will require the following math:

$$\begin{aligned} y(t) &= \int_0^t 1d\tau \\ &= t \end{aligned} \quad (4)$$

The third region,  $1 \leq t < 2$ , is solved in much the same manner. Take note of the changes in our integration though. As we move  $h(t - \tau)$  across our other function, the left-hand edge of the function,  $t - 1$ , becomes our lowlimit for the integral. This is shown through our convolution integral as

$$\begin{aligned} y(t) &= \int_{t-1}^1 1d\tau \\ &= 1 - (t - 1) \\ &= 2 - t \end{aligned} \quad (5)$$

The above formulas show the method for calculating convolution; however, do not let the simplicity of this example confuse you when you work on other problems. The method will be the same, you will just have to deal with more math in more complicated integrals.

#### 6.4 Convolution Results

Thus, we have the following results for our four regions:

$$y(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t < 1 \\ 2 - t & \text{if } 1 \leq t < 2 \\ 0 & \text{if } t \geq 2 \end{cases} \quad (6)$$

Now that we have found the resulting function for each of the four regions, we can combine them together and graph the convolution of  $x(t) * h(t)$ .

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**Figure 8:** Shows the system's response to the input,  $x(t)$ .

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