Continuous Time Fourier Transform (CTFT)*

Richard Baraniuk
Melissa Selik

Abstract

Details the Continuous-Time Fourier Transform.

1 Introduction

In this module, we will derive an expansion for any arbitrary continuous-time function, and in doing so, derive the **Continuous Time Fourier Transform** (CTFT).

Since complex exponentials are eigenfunctions of linear time-invariant (LTI) systems, calculating the output of an LTI system $\mathcal{H}$ given e^{st} as an input amounts to simple multiplication, where $H(s) \in \mathbb{C}$ is the eigenvalue corresponding to s. As shown in the figure, a simple exponential input would yield the output

$$y(t) = H(s)\, e^{st} \tag{1}$$

Using this and the fact that $\mathcal{H}$ is linear, calculating $y(t)$ for combinations of complex exponentials is also straightforward.

$$c_1 e^{s_1 t} + c_2 e^{s_2 t} \rightarrow c_1 H(s_1)\, e^{s_1 t} + c_2 H(s_2)\, e^{s_2 t}$$

$$\sum_n c_n e^{s_n t} \rightarrow \sum_n c_n H(s_n)\, e^{s_n t}$$

The action of H on an input such as those in the two equations above is easy to explain. $\mathcal{H}$ **independently scales** each exponential component $e^{s_n t}$ by a different complex number $H(s_n) \in \mathbb{C}$. As such, if we can write a function $f(t)$ as a combination of complex exponentials it allows us to easily calculate the output of a system.

Now, we will look to use the power of complex exponentials to see how we may represent arbitrary signals in terms of a set of simpler functions by superposition of a number of complex exponentials. Below we will

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present the **Continuous-Time Fourier Transform** (CTFT), commonly referred to as just the Fourier Transform (FT). Because the CTFT deals with nonperiodic signals, we must find a way to include all real frequencies in the general equations. For the CTFT we simply utilize integration over real numbers rather than summation over integers in order to express the aperiodic signals.

2 Fourier Transform Synthesis

Joseph Fourier[1] demonstrated that an arbitrary $s(t)$ can be written as a linear combination of harmonic complex sinusoids

$$s(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 nt} \tag{2}$$

where $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency. For almost all $s(t)$ of practical interest, there exists c_n to make (2) true. If $s(t)$ is finite energy ($s(t) \in L^2[0,T]$), then the equality in (2) holds in the sense of energy convergence; if $s(t)$ is continuous, then (2) holds pointwise. Also, if $s(t)$ meets some mild conditions (the Dirichlet conditions), then (2) holds pointwise everywhere except at points of discontinuity.

The c_n - called the Fourier coefficients - tell us "how much" of the sinusoid $e^{j\omega_0 nt}$ is in $s(t)$. The formula shows $s(t)$ as a sum of complex exponentials, each of which is easily processed by an LTI system (since it is an eigenfunction of **every** LTI system). Mathematically, it tells us that the set of complex exponentials $\left\{\forall n, n \in \mathbb{Z} : \left(e^{j\omega_0 nt}\right)\right\}$ form a basis for the space of T-periodic continuous time functions.

2.1 Equations

Now, in order to take this useful tool and apply it to arbitrary non-periodic signals, we will have to delve deeper into the use of the superposition principle. Let $s_T(t)$ be a periodic signal having period T. We want to consider what happens to this signal's spectrum as the period goes to infinity. We denote the spectrum for any assumed value of the period by $c_n(T)$. We calculate the spectrum according to the Fourier formula for a periodic signal, known as the Fourier Series (for more on this derivation, see the section on **Fourier Series**.)

$$c_n = \frac{1}{T}\int_0^T s(t)\, exp(-ß\omega_0 t)\ dt \tag{3}$$

where $\omega_0 = \frac{2\pi}{T}$ and where we have used a symmetric placement of the integration interval about the origin for subsequent derivational convenience. We vary the frequency index n proportionally as we increase the period. Define

$$S_T(f) \equiv Tc_n = \frac{1}{T}\int_0^T (S_T(f)\, exp(ß\omega_0 t)\ dt(4)$$

making the corresponding Fourier Series

$$s_T(t) = \sum_{-\infty}^{\infty} f(t)\, exp(ß\omega_0 t)\ \frac{1}{T} \tag{5}$$

As the period increases, the spectral lines become closer together, becoming a continuum. Therefore,

$$\lim_{T\to\infty} s_T(t) \equiv s(t) = \int_{-\infty}^{\infty} S(f)\, exp(ß\omega_0 t)\ df \tag{6}$$

with

$$S(f) = \int_{-\infty}^{\infty} s(t)\, exp(-ß\omega_0 t)\ dt \tag{7}$$

[1] http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

Continuous-Time Fourier Transform

$$\mathcal{F}(\Omega) = \int_{-\infty}^{\infty} f(t)\, e^{-(i\Omega t)} dt \quad (8)$$

Inverse CTFT

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\Omega)\, e^{i\Omega t} d\Omega \quad (9)$$

WARNING: It is not uncommon to see the above formula written slightly different. One of the most common differences is the way that the exponential is written. The above equations use the radial frequency variable Ω in the exponential, where $\Omega = 2\pi f$, but it is also common to include the more explicit expression, $i2\pi ft$, in the exponential. Click here for an overview of the notation used in Connexion's DSP modules.

Example 1

We know from Euler's formula that $cos(\omega t) + sin(\omega t) = \frac{1-j}{2} e^{j\omega t} + \frac{1+j}{2} e^{-j\omega t}$.

3 CTFT Definition Demonstration

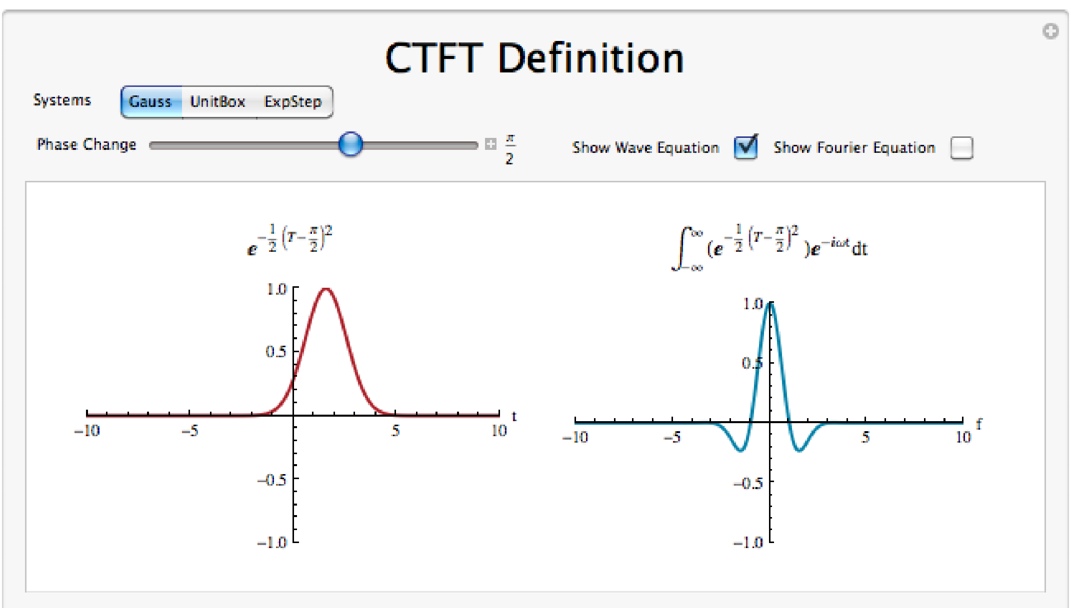

Figure 1: Interact (when online) with a Mathematica CDF demonstrating Continuous Time Fourier Transform. To Download, right-click and save as .cdf.

4 Example Problems

Exercise 1 ***(Solution on p. 5.)***

Find the Fourier Transform (CTFT) of the function

$$f(t) = \begin{cases} e^{-(\alpha t)} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

Exercise 2 ***(Solution on p. 5.)***

Find the inverse Fourier transform of the ideal lowpass filter defined by

$$X(\Omega) = \begin{cases} 1 & \text{if } |\Omega| \leq M \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

5 Fourier Transform Summary

Because complex exponentials are eigenfunctions of LTI systems, it is often useful to represent signals using a set of complex exponentials as a basis. The continuous time Fourier series synthesis formula expresses a continuous time, periodic function as the sum of continuous time, discrete frequency complex exponentials.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 nt} \quad (12)$$

The continuous time Fourier series analysis formula gives the coefficients of the Fourier series expansion.

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-(j\omega_0 nt)} dt \quad (13)$$

In both of these equations $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency.

Solutions to Exercises in this Module

Solution to Exercise (p. 4)

In order to calculate the Fourier transform, all we need to use is (8) (Continuous-Time Fourier Transform), complex exponentials, and basic calculus.

$$\begin{aligned} \mathcal{F}(\Omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-(i\Omega t)} dt \\ &= \int_{0}^{\infty} e^{-(\alpha t)} e^{-(i\Omega t)} dt \\ &= \int_{0}^{\infty} e^{(-t)(\alpha + i\Omega)} dt \\ &= 0 - \frac{-1}{\alpha + i\Omega} \end{aligned} \quad (14)$$

$$\mathcal{F}(\Omega) = \frac{1}{\alpha + i\Omega} \quad (15)$$

Solution to Exercise (p. 4)

Here we will use (9) (Inverse CTFT) to find the inverse FT given that $t \neq 0$.

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-M}^{M} e^{i(\Omega,t)} d\Omega \\ &= \frac{1}{2\pi} e^{i(\Omega,t)}|_{\Omega,\Omega=e^{iw}} \\ &= \frac{1}{\pi t} \sin(Mt) \end{aligned} \quad (16)$$

$$x(t) = \frac{M}{\pi} \left(\text{sinc} \frac{Mt}{\pi} \right) \quad (17)$$