Properties of the CTFT[*]

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Abstract

Describes in detail some properties of the Continuous Time Fourier Transform.

1 Introduction

This module will look at some of the basic properties of the Continuous-Time Fourier Transform[1] (CTFT).

NOTE: We will be discussing these properties for aperiodic, continuous-time signals but understand that very similar properties hold for discrete-time signals and periodic signals as well.

2 Discussion of Fourier Transform Properties

2.1 Linearity

The combined addition and scalar multiplication properties in the table above demonstrate the basic property of linearity. What you should see is that if one takes the Fourier transform of a linear combination of signals then it will be the same as the linear combination of the Fourier transforms of each of the individual signals. This is crucial when using a table[2] of transforms to find the transform of a more complicated signal.

Example 1

We will begin with the following signal:

$$z(t) = af_1(t) + bf_2(t) \tag{1}$$

Now, after we take the Fourier transform, shown in the equation below, notice that the linear combination of the terms is unaffected by the transform.

$$Z(\omega) = aF_1(\omega) + bF_2(\omega) \tag{2}$$

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[†]http://creativecommons.org/licenses/by/1.0
[1]"Continuous Time Fourier Transform (CTFT)" <http://cnx.org/content/m10098/latest/>
[2]"Common Fourier Transforms" <http://cnx.org/content/m10099/latest/>

2.2 Symmetry

Symmetry is a property that can make life quite easy when solving problems involving Fourier transforms. Basically what this property says is that since a rectangular function in time is a sinc function in frequency, then a sinc function in time will be a rectangular function in frequency. This is a direct result of the similarity between the forward CTFT and the inverse CTFT. The only difference is the scaling by 2π and a frequency reversal.

2.3 Time Scaling

This property deals with the effect on the frequency-domain representation of a signal if the time variable is altered. The most important concept to understand for the time scaling property is that signals that are narrow in time will be broad in frequency and *vice versa*. The simplest example of this is a delta function, a unit pulse[3] with a **very** small duration, in time that becomes an infinite-length constant function in frequency.

The table above shows this idea for the general transformation from the time-domain to the frequency-domain of a signal. You should be able to easily notice that these equations show the relationship mentioned previously: if the time variable is increased then the frequency range will be decreased.

2.4 Time Shifting

Time shifting shows that a shift in time is equivalent to a linear phase shift in frequency. Since the frequency content depends only on the shape of a signal, which is unchanged in a time shift, then only the phase spectrum will be altered. This property is proven below:

Example 2

We will begin by letting $z(t) = f(t-\tau)$. Now let us take the Fourier transform with the previous expression substituted in for $z(t)$.

$$Z(\omega) = \int_{-\infty}^{\infty} f(t-\tau)\, e^{-(i\omega t)}\, dt \tag{3}$$

Now let us make a simple change of variables, where $\sigma = t - \tau$. Through the calculations below, you can see that only the variable in the exponential are altered thus only changing the phase in the frequency domain.

$$\begin{aligned} Z(\omega) &= \int_{-\infty}^{\infty} f(\sigma)\, e^{-(i\omega(\sigma+\tau)t)}\, d\tau \\ &= e^{-(i\omega\tau)} \int_{-\infty}^{\infty} f(\sigma)\, e^{-(i\omega\sigma)}\, d\sigma \\ &= e^{-(i\omega\tau)} F(\omega) \end{aligned} \tag{4}$$

2.5 Convolution

Convolution is one of the big reasons for converting signals to the frequency domain, since convolution in time becomes multiplication in frequency. This property is also another excellent example of symmetry between time and frequency. It also shows that there may be little to gain by changing to the frequency domain when multiplication in time is involved.

[3]"Elemental Signals": Section Pulse <http://cnx.org/content/m0004/latest/#pulsedef>

We will introduce the convolution integral here, but if you have not seen this before or need to refresh your memory, then look at the continuous-time convolution[4] module for a more in depth explanation and derivation.

$$\begin{aligned} y(t) &= (f_1(t), f_2(t)) \\ &= \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau \end{aligned} \tag{5}$$

2.6 Time Differentiation

Since LTI[5] systems can be represented in terms of differential equations, it is apparent with this property that converting to the frequency domain may allow us to convert these complicated differential equations to simpler equations involving multiplication and addition. This is often looked at in more detail during the study of the Laplace Transform[6].

2.7 Parseval's Relation

$$\int_{-\infty}^{\infty} (|f(t)|)^2 dt = \int_{-\infty}^{\infty} (|F(\omega)|)^2 df \tag{6}$$

Parseval's relation tells us that the energy of a signal is equal to the energy of its Fourier transform.

Figure 1

2.8 Modulation (Frequency Shift)

Modulation is absolutely imperative to communications applications. Being able to shift a signal to a different frequency, allows us to take advantage of different parts of the electromagnetic spectrum is what allows us to transmit television, radio and other applications through the same space without significant interference.

The proof of the frequency shift property is very similar to that of the time shift (Section 2.4: Time Shifting); however, here we would use the inverse Fourier transform in place of the Fourier transform. Since we went through the steps in the previous, time-shift proof, below we will just show the initial and final step to this proof:

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega - \phi) e^{i\omega t} d\omega \tag{7}$$

Now we would simply reduce this equation through another change of variables and simplify the terms. Then we will prove the property expressed in the table above:

$$z(t) = f(t) e^{i\phi t} \tag{8}$$

[4]"Continuous Time Convolution" <http://cnx.org/content/m10085/latest/>
[5]"System Classifications and Properties" <http://cnx.org/content/m10084/latest/>
[6]"The Laplace Transform" <http://cnx.org/content/m10110/latest/>

3 Properties Demonstration

An interactive example demonstration of the properties is included below:

This media object is a LabVIEW VI. Please view or download it at
<http://cnx.org/content/m10100/2.15/CTFTSPlab.llb>

Figure 2: Interactive Signal Processing Laboratory Virtual Instrument created using NI's Labview.

4 Summary Table of CTFT Properties

Operation Name	**Signal (** $f(t)$ **)**	**Transform (** $F(\omega)$ **)**
Linearity (Section 2.1: Linearity)	$a(f_1, t) + b(f_2, t)$	$a(F_1, \omega) + b(F_2, \omega)$
Scalar Multiplication (Section 2.1: Linearity)	$\alpha f(t)$	$\alpha F(\omega)$
Symmetry (Section 2.2: Symmetry)	$F(t)$	$2\pi f(-\omega)$
Time Scaling (Section 2.3: Time Scaling)	$f(\alpha t)$	$\frac{1}{\vert\alpha\vert} F\left(\frac{\omega}{\alpha}\right)$
Time Shift (Section 2.4: Time Shifting)	$f(t-\tau)$	$F(\omega) e^{-(i\omega\tau)}$
Convolution in Time (Section 2.5: Convolution)	$(f_1(t), f_2(t))$	$F_1(t) F_2(t)$
Convolution in Frequency (Section 2.5: Convolution)	$f_1(t) f_2(t)$	$\frac{1}{2\pi}(F_1(t), F_2(t))$
Differentiation (Section 2.6: Time Differentiation)	$\frac{d^n}{dt^n} f(t)$	$(i\omega)^n F(\omega)$
Parseval's Theorem (Section 2.7: Parseval's Relation)	$\int_{-\infty}^{\infty} (\vert f(t)\vert)^2 dt$	$\int_{-\infty}^{\infty} (\vert F(\omega)\vert)^2 df$
Modulation (Frequency Shift) (Section 2.8: Modulation (Frequency Shift))	$f(t) e^{i\phi t}$	$F(\omega - \phi)$

Table 1: Table of Fourier Transform Properties