Butterworth Filters*

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Abstract

Describes the design of analog lowpass Butterworth filters.

The Butterworth filter is a filter that can be constructed out of passive R, L, C circuits. The magnitude of the transfer function for this filter is

Magnitude of Butterworth Filter Transfer Function

$$|H(i\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}} \quad (1)$$

where n is the **order** of the filter and ω_c is the **cutoff frequency**. The cutoff frequency is the frequency where the magnitude experiences a 3 dB dropoff (where $|H(i\omega)| = \frac{1}{\sqrt{2}}$).

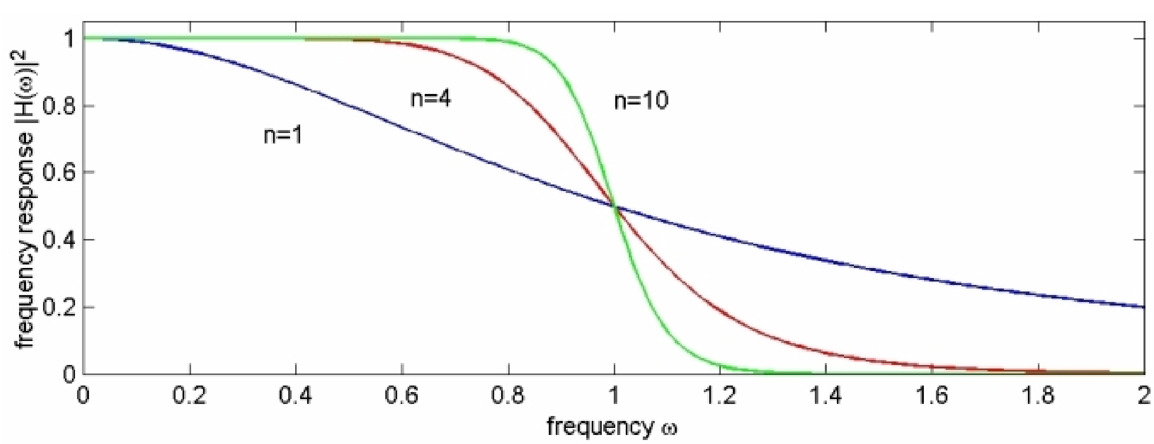

Figure 1: Three different orders of lowpass Butterworth analog filters: $n = \{1, 4, 10\}$. As n increases, the filter more closely approximates an ideal brickwall lowpass response.

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The important aspects of Figure 1 are that it does not ripple in the passband or stopband as other filters tend to, and that the larger n, the sharper the cutoff (the smaller the transition band[1]).

This transfer function is often seen in its normalized form of

Magnitude of Normalized Transfer Function for Lowpass Butterworth Filter

$$|H(i\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}} \quad (2)$$

Butterworth filters give transfer functions ($H(i\omega)$ and $H(s)$) that are **rational functions**. They also have only poles[2], resulting in a transfer function of the form

$$\frac{1}{(s-s_1)(s-s_2)\cdots(s-s_n)} \quad (3)$$

and a pole-zero plot of

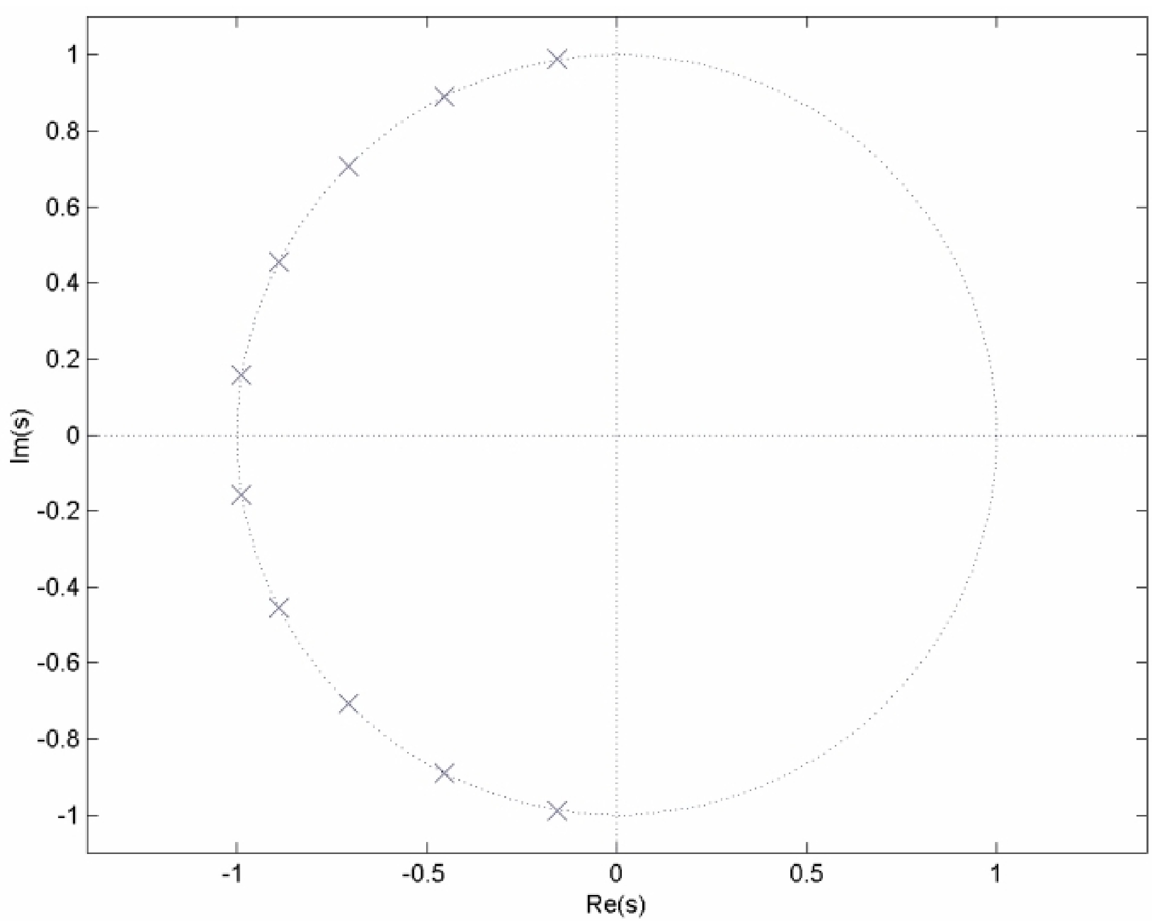

Figure 2: Poles of a 10th-order ($n = 5$) lowpass Butterworth filter.

[1]"Practical Filters" <http://cnx.org/content/m10126/latest/>
[2]"Poles and Zeros" <http://cnx.org/content/m10112/latest/>

Note that the poles lie along a circle in the s-plane.

1 Designing a Butterworth Filter

Designing a Butterworth filter is a trivial task. Since we know that the filter contains only poles, we know that we can write it as

$$H(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1} \quad (4)$$

From this, we may look up the a_i from a table (like the one below) for any desired n. We can also find them in Matlab by using the `buttap` command. The real challenge of designing a Butterworth filter comes with figuring out the optimal characteristics for the given application.

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	1.414214								
3	2.000000	2.000000							
4	2.613126	3.414214	2.613126						
5	3.236068	5.236068	5.236068	3.236068					
6	3.863703	7.464102	9.141620	7.464102	3.863703				
7	4.493959	10.097835	14.591794	14.591794	10.097835	4.493959			
8	5.125831	13.137071	21.846151	25.688356	21.846151	13.137071	5.125831		
9	5.758770	16.581719	31.163437	41.986386	41.986386	31.163437	16.581719	5.758770	
10	6.392453	20.431729	42.802061	64.882396	74.233429	64.882396	42.802061	20.431729	6.392453

Table 1

Exercise 1 ***(Solution on p. 4.)***

Design a Butterworth filter with a passband gain between 1 and 0.891 (-1 dB gain) for $0 < \omega < 10$ and a stopband not to exceed 0.0316 (-30 dB gain) for $\omega \geq 20$.

Solutions to Exercises in this Module

Solution to Exercise (p. 3)

The first step is to determine n. To do this, we must solve for n using the passband and stopband criteria. We begin by finding the equation for the gain in the passband in dB,

$$\begin{aligned} \hat{G_p} &= 20\log\left|H\left(i\omega\right)\right| \\ &= -10\log\left(1+\left(\frac{\omega_p}{\omega_c}\right)^{2n}\right) \end{aligned} \tag{5}$$

and for the stopband in dB,

$$\begin{aligned} \hat{G_s} &= 20\log\left|H\left(i\omega\right)\right| \\ &= -10\log\left(1+\left(\frac{\omega_s}{\omega_c}\right)^{2n}\right) \end{aligned} \tag{6}$$

these equations can also take the form

$$\left(\frac{\omega_x}{\omega_c}\right)^{2n} = 10^{\frac{-\hat{G_x}}{10}} - 1 \tag{7}$$

In this form, we may divide the passband equation by the stopband equation to get rid of the ω_c. From there, we can solve for n to get

$$n = \frac{\log\frac{10^{\frac{-\hat{G_s}}{10}}-1}{10^{\frac{-\hat{G_p}}{10}}-1}}{2\log\frac{\omega_s}{\omega_p}} \tag{8}$$

By plugging in, we find $n = 5.9569$. However, since n must be an integer, we round this up to $n = 6$

The next step is to find ω_c. We can do this by substituting $n = 6$ into the equations for the passband and stopband and solving for ω_c. This yields $\omega_c = 11.1919$ for the passband equation and $\omega_c = 11.2478$ for the stopband equation. The difference in these solutions is a result of n needing to be an integer. If we choose the solution from the passband equation, the passband will meet its requirements exactly, and the stopband will surpass its requirements. If we choose the solution from the stopband equation instead, the stopband requirements will be met exactly, while we will exceed the passband requirements. Therefore, we may choose either value or any value in between. For this example, we will choose $\omega_c = 11.2478$.

Now, we can find the normalized transfer function. Since we know this to be a sixth-order Butterworth, we can determine from the table that

$$H\left(s\right) = \frac{1}{s^6 + 3.863703s^5 + 7.464102s^4 + 9.141620s^3 + 7.464102s^2 + 3.863703s + 1} \tag{9}$$

Finally, we can determine the final transfer function.

$$H\left(s\right) = \frac{1}{\left(\frac{s}{11.2478}\right)^6 + 3.863703\left(\frac{s}{11.2478}\right)^5 + 7.464102\left(\frac{s}{11.2478}\right)^4 + 9.141620\left(\frac{s}{11.2478}\right)^3 + 7.464102\left(\frac{s}{11.2478}\right)^2 + 3.863703\frac{s}{11.2478} + 1} \tag{10}$$

Rather than multiplying this out and factoring, we will leave it in this form for readability, since the numbers can get quite large otherwise.