Introduction to Stochastic Processes*

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Abstract

Describes signals that cannot be precisely characterized.

1 Definitions, distributions, and stationarity

Definition 1: Stochastic Process
Given a sample space, a stochastic process is an indexed collection of random variables defined for each $\omega \in \Omega$.

$$\forall t, t \in \mathbb{R} : (X_t(\omega)) \tag{1}$$

Example 1
Received signal at an antenna as in Figure 1.

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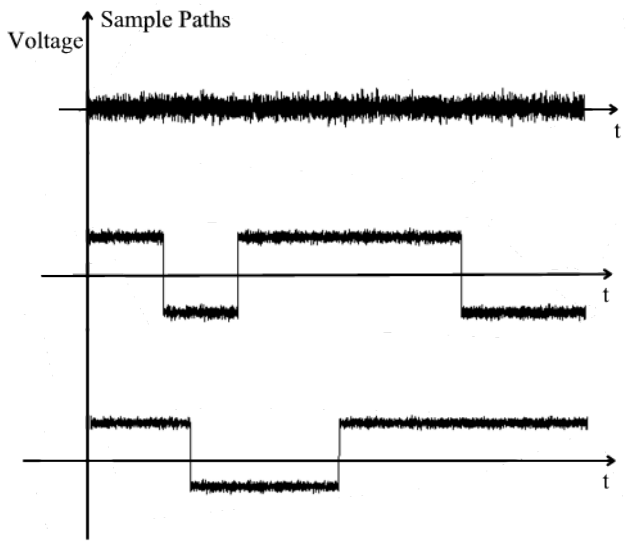

Figure 1

For a given t, $X_t(\omega)$ is a random variable with a distribution

First-order distribution

$$\begin{aligned} F_{X_t}(b) &= Pr[X_t \leq b] \\ &= Pr[\{\omega \in \Omega \mid X_t(\omega) \leq b\}] \end{aligned} \tag{2}$$

Definition 2: First-order stationary process

If $F_{X_t}(b)$ is not a function of time then X_t is called a first-order stationary process.

Second-order distribution

$$F_{X_{t_1},X_{t_2}}(b_1,b_2) = Pr[X_{t_1} \leq b_1, X_{t_2} \leq b_2] \tag{3}$$

for all $t_1 \in \mathbb{R}$, $t_2 \in \mathbb{R}$, $b_1 \in \mathbb{R}$, $b_2 \in \mathbb{R}$

Nth-order distribution

$$F_{X_{t_1},X_{t_2},\ldots,X_{t_N}}(b_1,b_2,\ldots,b_N) = Pr[X_{t_1} \leq b_1,\ldots,X_{t_N} \leq b_N] \tag{4}$$

Nth-order stationary : A random process is stationary of order N if

$$F_{X_{t_1},X_{t_2},\ldots,X_{t_N}}(b_1,b_2,\ldots,b_N) = F_{X_{t_1+T},X_{t_2+T},\ldots,X_{t_N+T}}(b_1,b_2,\ldots,b_N) \tag{5}$$

Strictly stationary : A process is strictly stationary if it is Nth order stationary for all N.

Example 2

$X_t = \cos(2\pi f_0 t + \Theta(\omega))$ where f_0 is the deterministic carrier frequency and $\Theta(\omega) : \Omega \to \mathbb{R}$ is a random variable defined over $[-\pi,\pi]$ and is assumed to be a uniform random variable; *i.e.*,

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi} & \text{if } \theta \in [-\pi, \pi] \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} F_{X_t}(b) &= Pr[X_t \leq b] \\ &= Pr[\cos(2\pi f_0 t + \Theta) \leq b] \end{aligned} \tag{6}$$

$$F_{X_t}(b) = Pr[-\pi \leq 2\pi f_0 t + \Theta \leq -\text{arccos}(b)] + Pr[\text{arccos}(b) \leq 2\pi f_0 t + \Theta \leq \pi] \tag{7}$$

$$\begin{aligned} F_{X_t}(b) &= \int_{(-\pi)-2\pi f_0 t}^{(-\text{arccos}(b))-2\pi f_0 t} \frac{1}{2\pi} d\theta + \int_{\text{arccos}(b)-2\pi f_0 t}^{\pi-2\pi f_0 t} \frac{1}{2\pi} d\theta \\ &= (2\pi - 2\text{arccos}(b)) \frac{1}{2\pi} \end{aligned} \tag{8}$$

$$\begin{aligned} f_{X_t}(x) &= \frac{d}{dx}\left(1 - \frac{1}{\pi}\text{arccos}(x)\right) \\ &= \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{9}$$

This process is stationary of order 1.

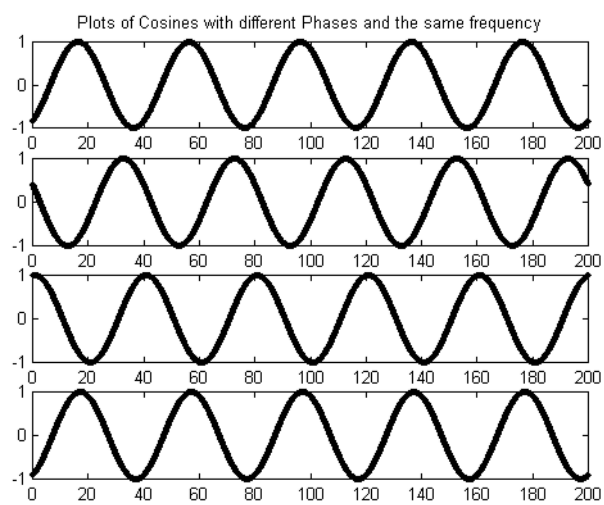

Figure 2

The second order stationarity can be determined by first considering conditional densities and the joint density. Recall that

$$X_t = \cos(2\pi f_0 t + \Theta) \tag{10}$$

Then the relevant step is to find

$$Pr[X_{t_2} \leq b_2 \mid X_{t_1} = x_1] \tag{11}$$

Note that

$$(X_{t_1} = x_1 = \cos(2\pi f_0 t + \Theta)) \Rightarrow (\Theta = \arccos(x_1) - 2\pi f_0 t) \tag{12}$$

$$\begin{aligned} X_{t_2} &= \cos(2\pi f_0 t_2 + \arccos(x_1) - 2\pi f_0 t_1) \\ &= \cos(2\pi f_0 (t_2 - t_1) + \arccos(x_1)) \end{aligned} \tag{13}$$

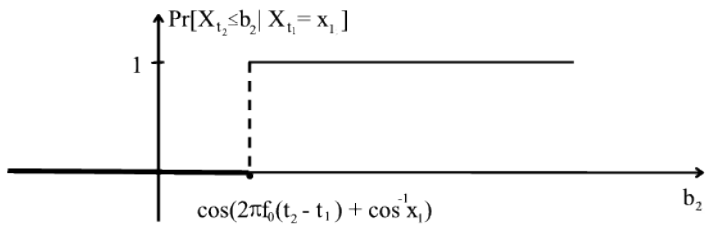

Figure 3

$$F_{X_{t_2},X_{t_1}}(b_2, b_1) = \int_{-\infty}^{b_1} f_{X_{t_1}}(x_1) \, Pr[X_{t_2} \leq b_2 \mid X_{t_1} = x_1] \, d\text{x}1 \tag{14}$$

Note that this is only a function of $t_2 - t_1$.

Example 3

Every T seconds, a fair coin is tossed. If heads, then $X_t = 1$ for $nT \leq t < (n+1)T$. If tails, then $X_t = -1$ for $nT \leq t < (n+1)T$.

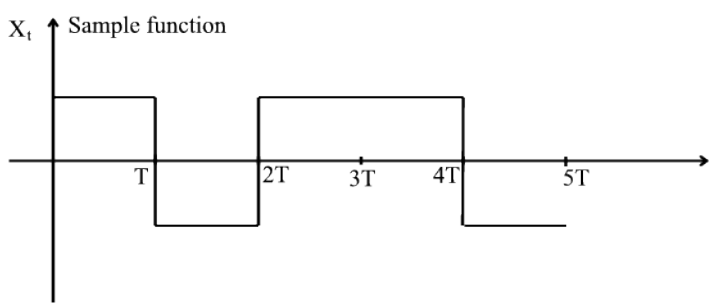

Figure 4

$$p_{X_t}(x) = \begin{cases} \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = -1 \end{cases} \tag{15}$$

for all $t \in \mathbb{R}$. X_t is stationary of order 1.

Second order probability mass function

$$p_{X_{t_1}X_{t_2}}\left(x_1,x_2\right)=p_{X_{t_2}|X_{t_1}}\left(x_2|x_1\right)p_{X_{t_1}}\left(x_1\right) \tag{16}$$

The conditional pmf

$$p_{X_{t_2}|X_{t_1}}\left(x_2|x_1\right)=\begin{cases} 0 & \text{if} \quad x_2\neq x_1 \\ 1 & \text{if} \quad x_2=x_1 \end{cases} \tag{17}$$

when $nT\leq t_1<\left(n+1\right)T$ and $nT\leq t_2<\left(n+1\right)T$ for some n.

$$p_{X_{t_2}|X_{t_1}}\left(x_2|x_1\right)=p_{X_{t_2}}\left(x_2\right) \tag{18}$$

for all x_1 and for all x_2 when $nT\leq t_1<\left(n+1\right)T$ and $mT\leq t_2<\left(m+1\right)T$ with $n\neq m$

$$p_{X_{t_2}X_{t_1}}\left(x_2,x_1\right)=\begin{cases} 0 \quad \text{if} \quad x_2\neq x_1 \text{for } nT\leq t_1,t_2<\left(n+1\right)T \\ p_{X_{t_1}}\left(x_1\right) \quad \text{if} \quad x_2=x_1 \text{for } nT\leq t_1,t_2<\left(n+1\right)T \\ p_{X_{t_1}}\left(x_1\right)p_{X_{t_2}}\left(x_2\right) \quad \text{if} \quad n\neq m \text{for } \left(nT\leq t_1<\left(n+1\right)T\right) \ \wedge \ \left(mT\leq t_2<\left(m+1\right)T\right. \end{cases} \tag{19}$$