Second-order Description*

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Abstract

Describes signals that cannot be precisely characterized.

1 Second-order description

Practical and incomplete statistics

Definition 1: Mean

The mean function of a random process X_t is defined as the expected value of X_t for all t's.

$$\begin{aligned} \mu_{X_t} &= E[X_t] \\ &= \begin{cases} \int_{-\infty}^{\infty} x f_{X_t}(x)\,dx & \text{if continuous} \\ \sum_{k=-\infty}^{\infty} x_k p_{X_t}(x_k) & \text{if discrete} \end{cases} \end{aligned} \quad (1)$$

Definition 2: Autocorrelation

The autocorrelation function of the random process X_t is defined as

$$\begin{aligned} R_X(t_2, t_1) &= E\left[X_{t_2}\overline{X_{t_1}}\right] \\ &= \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 \overline{x_1} f_{X_{t_2}, X_{t_1}}(x_2, x_1)\,dx1dx2 & \text{if continuous} \\ \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} x_l \overline{x_k} p_{X_{t_2}, X_{t_1}}(x_l, x_k) & \text{if discrete} \end{cases} \end{aligned} \quad (2)$$

Rule 1:

If X_t is second-order stationary, then $R_X(t_2, t_1)$ only depends on $t_2 - t_1$.

Proof:

$$\begin{aligned} R_X(t_2, t_1) &= E\left[X_{t_2}\overline{X_{t_1}}\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 \overline{x_1} f_{X_{t_2}, X_{t_1}}(x_2, x_1)\,dx\,2dx\,1 \end{aligned} \quad (3)$$

$$\begin{aligned} R_X(t_2, t_1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 \overline{x_1} f_{X_{t_2 - t_1}, X_0}(x_2, x_1)\,dx\,2dx\,1 \\ &= R_X(t_2 - t_1, 0) \end{aligned} \quad (4)$$

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If $R_X(t_2,t_1)$ depends on t_2-t_1 only, then we will represent the autocorrelation with only one variable $\tau=t_2-t_1$

$$\begin{aligned} R_X(\tau) &= R_X(t_2-t_1) \\ &= R_X(t_2,t_1) \end{aligned} \quad (5)$$

Properties

1. $R_X(0)\geq 0$
2. $R_X(\tau)=\overline{R_X(-\tau)}$
3. $|R_X(\tau)|\leq R_X(0)$

Example 1

$X_t=\cos(2\pi f_0 t+\Theta(\omega))$ and Θ is uniformly distributed between 0 and 2π. The mean function

$$\begin{aligned} \mu_X(t) &= E[X_t] \\ &= E[\cos(2\pi f_0 t+\Theta)] \\ &= \int_0^{2\pi}\cos(2\pi f_0 t+\theta)\frac{1}{2\pi}d\theta \\ &= 0 \end{aligned} \quad (6)$$

The autocorrelation function

$$\begin{aligned} R_X(t+\tau,t) &= E\left[X_{t+\tau}\overline{X_t}\right] \\ &= E[\cos(2\pi f_0(t+\tau)+\Theta)\cos(2\pi f_0 t+\Theta)] \\ &= 1/2E[\cos(2\pi f_0\tau)]+1/2E[\cos(2\pi f_0(2t+\tau)+2\Theta)] \\ &= 1/2\cos(2\pi f_0\tau)+1/2\int_0^{2\pi}\cos(2\pi f_0(2t+\tau)+2\theta)\frac{1}{2\pi}d\theta \\ &= 1/2\cos(2\pi f_0\tau) \end{aligned} \quad (7)$$

Not a function of t since the second term in the right hand side of the equality in (7) is zero.

Example 2

Toss a fair coin every T seconds. Since X_t is a discrete valued random process, the statistical characteristics can be captured by the pmf and the mean function is written as

$$\begin{aligned} \mu_X(t) &= E[X_t] \\ &= 1/2\times -1+1/2\times 1 \\ &= 0 \end{aligned} \quad (8)$$

$$\begin{aligned} R_X(t_2,t_1) &= \sum_{kk}\sum_{ll}x_k x_l p_{X_{t_2},X_{t_1}}(x_k,x_l) \\ &= 1\times 1\times 1/2-1\times -1\times 1/2 \\ &= 1 \end{aligned} \quad (9)$$

when $nT\leq t_1<(n+1)T$ and $nT\leq t_2<(n+1)T$

$$\begin{aligned} R_X(t_2,t_1) &= 1\times 1\times 1/4-1\times -1\times 1/4-1\times 1\times 1/4+1\times -1\times 1/4 \\ &= 0 \end{aligned} \quad (10)$$

when $nT \leq t_1 < (n+1)T$ and $mT \leq t_2 < (m+1)T$ with $n \neq m$

$$R_X(t_2, t_1) = \begin{cases} 1 \text{ if } (nT \leq t_1 < (n+1)T) \wedge (nT \leq t_2 < (n+1)T) \\ 0 \text{ otherwise} \end{cases} \quad (11)$$

A function of t_1 and t_2.

Definition 3: Wide Sense Stationary

A process is said to be wide sense stationary if μ_X is constant and $R_X(t_2, t_1)$ is only a function of $t_2 - t_1$.

Rule 2:

If X_t is strictly stationary, then it is wide sense stationary. The converse is not necessarily true.

Definition 4: Autocovariance

Autocovariance of a random process is defined as

$$\begin{aligned} C_X(t_2, t_1) &= E\left[(X_{t_2} - \mu_X(t_2))\overline{X_{t_1} - \mu_X(t_1)}\right] \\ &= R_X(t_2, t_1) - \mu_X(t_2)\overline{\mu_X(t_1)} \end{aligned} \quad (12)$$

The variance of X_t is $\text{Var}(X_t) = C_X(t, t)$

Two processes defined on one experiment (Figure 1).

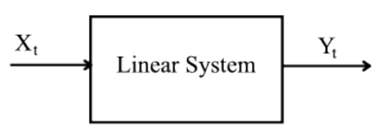

Figure 1

Definition 5: Crosscorrelation

The crosscorrelation function of a pair of random processes is defined as

$$\begin{aligned} R_{XY}(t_2, t_1) &= E\left[X_{t_2}\overline{Y_{t_1}}\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X_{t_2}, Y_{t_1}}(x, y)\, dx dy \end{aligned} \quad (13)$$

$$C_{XY}(t_2, t_1) = R_{XY}(t_2, t_1) - \mu_X(t_2)\overline{\mu_Y(t_1)} \quad (14)$$

Definition 6: Jointly Wide Sense Stationary

The random processes X_t and Y_t are said to be jointly wide sense stationary if $R_{XY}(t_2, t_1)$ is a function of $t_2 - t_1$ only and $\mu_X(t)$ and $\mu_Y(t)$ are constant.