# Discrete-Time Fourier Transform (DTFT)*

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**Abstract**

Discussion of Discrete-time Fourier Transforms. Topics include comparison with analog transforms and discussion of Parseval's theorem.

The Fourier transform of the discrete-time signal $s(n)$ is defined to be

$$S\left(e^{i2\pi f}\right) = \sum_{n=-\infty}^{\infty} s(n)\, e^{-(i2\pi f n)} \tag{1}$$

Frequency here has no units. As should be expected, this definition is linear, with the transform of a sum of signals equaling the sum of their transforms. Real-valued signals have conjugate-symmetric spectra: $S\left(e^{-(i2\pi f)}\right) = \overline{S\left(e^{j2\pi f}\right)}$.

**Exercise 1** *(Solution on p. 7.)*

A special property of the discrete-time Fourier transform is that it is periodic with period one: $S\left(e^{i2\pi(f+1)}\right) = S\left(e^{i2\pi f}\right)$. Derive this property from the definition of the DTFT.

Because of this periodicity, we need only plot the spectrum over one period to understand completely the spectrum's structure; typically, we plot the spectrum over the frequency range $\left[-\frac{1}{2}, \frac{1}{2}\right]$. When the signal is real-valued, we can further simplify our plotting chores by showing the spectrum only over $\left[0, \frac{1}{2}\right]$; the spectrum at negative frequencies can be derived from positive-frequency spectral values.

When we obtain the discrete-time signal via sampling an analog signal, the Nyquist frequency[1] corresponds to the discrete-time frequency $\frac{1}{2}$. To show this, note that a sinusoid having a frequency equal to the Nyquist frequency $\frac{1}{2T_s}$ has a sampled waveform that equals

$$\cos\left(2\pi \times \frac{1}{2Ts} nTs\right) = \cos(\pi n) = (-1)^n$$

The exponential in the DTFT at frequency $\frac{1}{2}$ equals $e^{-\frac{i2\pi n}{2}} = e^{-(i\pi n)} = (-1)^n$, meaning that discrete-time frequency equals analog frequency multiplied by the sampling interval

$$f_D = f_A T_s \tag{2}$$

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[1]"The Sampling Theorem" <http://cnx.org/content/m0050/latest/#para1>

$f_D$ and $f_A$ represent discrete-time and analog frequency variables, respectively. The aliasing figure[2] provides another way of deriving this result. As the duration of each pulse in the periodic sampling signal $p_{T_s}(t)$ narrows, the amplitudes of the signal's spectral repetitions, which are governed by the Fourier series coefficients[3] of $p_{T_s}(t)$, become increasingly equal. Examination of the periodic pulse signal[4] reveals that as $\Delta$ decreases, the value of $c_0$, the largest Fourier coefficient, decreases to zero: $|c_0| = \frac{A\Delta}{T_s}$. Thus, to maintain a mathematically viable Sampling Theorem, the amplitude $A$ must increase as $\frac{1}{\Delta}$, becoming infinitely large as the pulse duration decreases. Practical systems use a small value of $\Delta$, say $0.1 \cdot T_s$ and use amplifiers to rescale the signal. Thus, the sampled signal's spectrum becomes periodic with period $\frac{1}{T_s}$. Thus, the Nyquist frequency $\frac{1}{2T_s}$ corresponds to the frequency $\frac{1}{2}$.

**Example 1**

Let's compute the discrete-time Fourier transform of the exponentially decaying sequence $s(n) = a^n u(n)$, where $u(n)$ is the unit-step sequence. Simply plugging the signal's expression into the Fourier transform formula,

$$\begin{aligned} S\left(e^{i2\pi f}\right) &= \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-(i2\pi f n)} \\ &= \sum_{n=0}^{\infty} \left(a e^{-(i2\pi f)}\right)^n \end{aligned} \tag{3}$$

This sum is a special case of the **geometric series**.

$$\sum_{n=0}^{\infty} \alpha^n = \forall \alpha, |\alpha| < 1 : \left(\frac{1}{1-\alpha}\right) \tag{4}$$

Thus, as long as $|a| < 1$, we have our Fourier transform.

$$S\left(e^{i2\pi f}\right) = \frac{1}{1 - a e^{-(i2\pi f)}} \tag{5}$$

Using Euler's relation, we can express the magnitude and phase of this spectrum.

$$|S\left(e^{i2\pi f}\right)| = \frac{1}{\sqrt{(1 - a\cos(2\pi f))^2 + a^2 \sin^2(2\pi f)}} \tag{6}$$

$$\angle\left(S\left(e^{i2\pi f}\right)\right) = -\tan^{-1}\left(\frac{a\sin(2\pi f)}{1 - a\cos(2\pi f)}\right) \tag{7}$$

No matter what value of $a$ we choose, the above formulae clearly demonstrate the periodic nature of the spectra of discrete-time signals. Figure 1 (Spectrum of exponential signal) shows indeed that the spectrum is a periodic function. We need only consider the spectrum between $-\frac{1}{2}$ and $\frac{1}{2}$ to unambiguously define it. When $a > 0$, we have a lowpass spectrum—the spectrum diminishes as frequency increases from 0 to $\frac{1}{2}$—with increasing $a$ leading to a greater low frequency content; for $a < 0$, we have a highpass spectrum (Figure 2 (Spectra of exponential signals)).

[2]"The Sampling Theorem", Figure 2: aliasing <http://cnx.org/content/m0050/latest/#alias>
[3]"Complex Fourier Series", (10) <http://cnx.org/content/m0042/latest/#eqn2>
[4]"Complex Fourier Series", Figure 1 <http://cnx.org/content/m0042/latest/#pps>

**Spectrum of exponential signal**

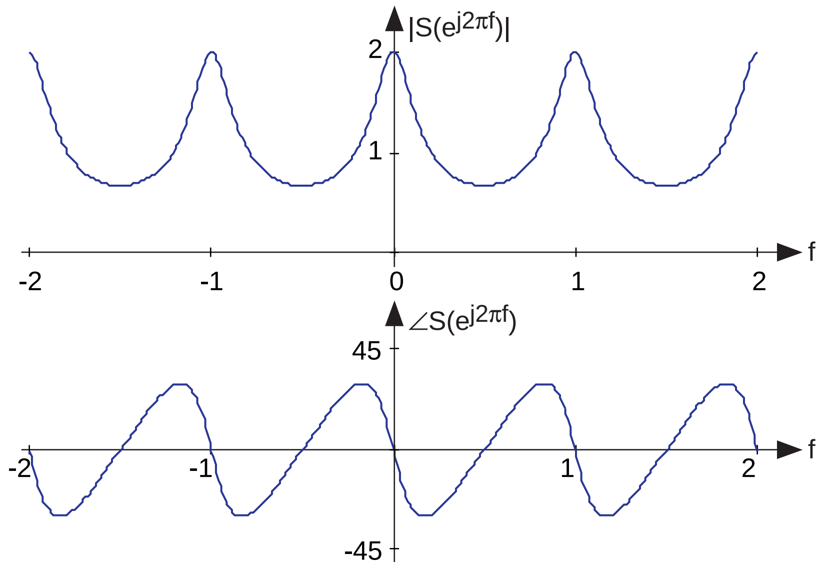


**Figure 1:** The spectrum of the exponential signal ($a = 0.5$) is shown over the frequency range [-2, 2], clearly demonstrating the periodicity of all discrete-time spectra. The angle has units of degrees.

**Spectra of exponential signals**

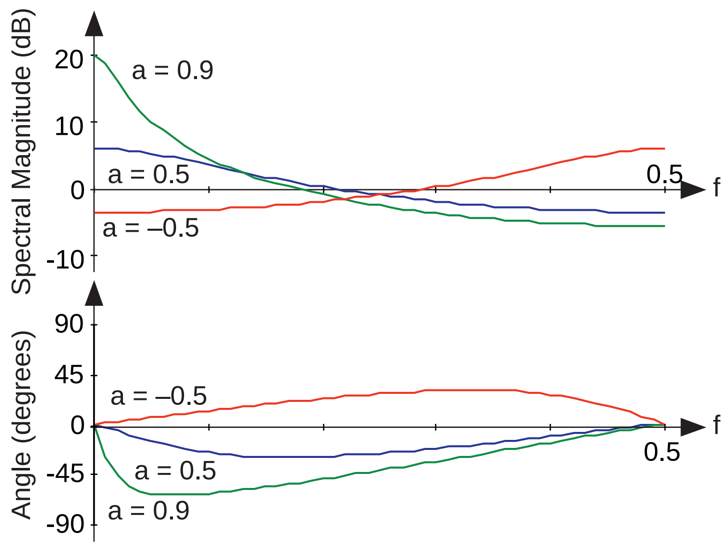


**Figure 2:** The spectra of several exponential signals are shown. What is the apparent relationship between the spectra for $a = 0.5$ and $a = -0.5$?

**Example 2**

Analogous to the analog pulse signal, let's find the spectrum of the length-$N$ pulse sequence.

$$s(n) = \begin{cases} 1 & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

The Fourier transform of this sequence has the form of a truncated geometric series.

$$S\left(e^{i2\pi f}\right) = \sum_{n=0}^{N-1} e^{-(i2\pi fn)} \quad (9)$$

For the so-called finite geometric series, we know that

$$\sum_{n=n_0}^{N+n_0-1} \alpha^n = \alpha^{n_0} \frac{1-\alpha^N}{1-\alpha} \quad (10)$$

for **all** values of $\alpha$.

**Exercise 2** ***(Solution on p. 7.)***

Derive this formula for the finite geometric series sum. The "trick" is to consider the difference between the series' sum and the sum of the series multiplied by $\alpha$.

Applying this result yields (Figure 3 (Spectrum of length-ten pulse).)

$$\begin{aligned} S\left(e^{i2\pi f}\right) &= \frac{1-e^{-(i2\pi fN)}}{1-e^{-(i2\pi f)}} \\ &= e^{-(i\pi f(N-1))} \frac{\sin(\pi fN)}{\sin(\pi f)} \end{aligned} \quad (11)$$

The ratio of sine functions has the generic form of $\frac{\sin(Nx)}{\sin(x)}$, which is known as the **discrete-time sinc function**$\mathrm{dsinc}(x)$. Thus, our transform can be concisely expressed as $S\left(e^{i2\pi f}\right) = e^{-(i\pi f(N-1))}\mathrm{dsinc}(\pi f)$. The discrete-time pulse's spectrum contains many ripples, the number of which increase with $N$, the pulse's duration.

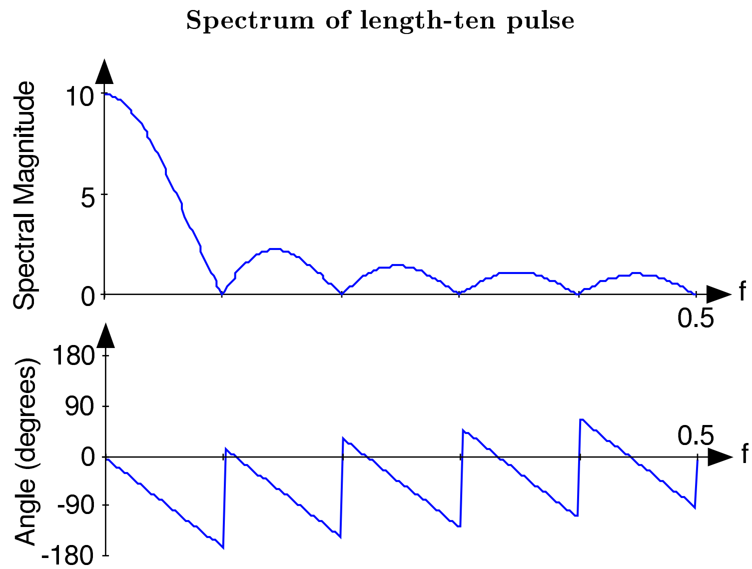


**Figure 3:** The spectrum of a length-ten pulse is shown. Can you explain the rather complicated appearance of the phase?

The inverse discrete-time Fourier transform is easily derived from the following relationship:

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-(i2\pi fm)} e^{i2\pi fn} df &= \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \\ &= \delta(m-n)\end{aligned} \tag{12}$$

Therefore, we find that

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} S\left(e^{i2\pi f}\right) e^{i2\pi fn} df &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{mm} s(m)\, e^{-(i2\pi fm)} e^{i2\pi fn} df \\ &= \sum_{mm} s(m) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{(-(i2\pi f))(m-n)} df \\ &= s(n)\end{aligned} \tag{13}$$

The Fourier transform pairs in discrete-time are

$$\begin{aligned}&S\left(e^{i2\pi f}\right) = \sum_{n=-\infty}^{\infty} s(n)\, e^{-(i2\pi fn)} \\ &s(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S\left(e^{i2\pi f}\right) e^{i2\pi fn} df\end{aligned} \tag{14}$$

The properties of the discrete-time Fourier transform mirror those of the analog Fourier transform. The DTFT properties table [5] shows similarities and differences. One important common property is Parseval's Theorem.

$$\sum_{n=-\infty}^{\infty} \left(|s(n)|\right)^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(|S\left(e^{i2\pi f}\right)|\right)^2 df \tag{15}$$

To show this important property, we simply substitute the Fourier transform expression into the frequency-domain expression for power.

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(|S\left(e^{i2\pi f}\right)|\right)^2 df &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{nn} s(n)\, e^{-(i2\pi f n)} \sum_{mm} \overline{s(n)} e^{i2\pi f m} df \\ &= \sum_{\mathrm{n,m}n,m} s(n)\, \overline{s(n)} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi f(m-n)} df \end{aligned} \tag{16}$$

Using the orthogonality relation (12), the integral equals $\delta(m-n)$, where $\delta(n)$ is the unit sample[6]. Thus, the double sum collapses into a single sum because nonzero values occur only when $n=m$, giving Parseval's Theorem as a result. We term $\sum_{nn} s^2(n)$ the energy in the discrete-time signal $s(n)$ in spite of the fact that discrete-time signals don't consume (or produce for that matter) energy. This terminology is a carry-over from the analog world.

**Exercise 3** *(Solution on p. 7.)*

Suppose we obtained our discrete-time signal from values of the product $s(t)\, p_{T_s}(t)$, where the duration of the component pulses in $p_{T_s}(t)$ is $\Delta$. How is the discrete-time signal energy related to the total energy contained in $s(t)$? Assume the signal is bandlimited and that the sampling rate was chosen appropriate to the Sampling Theorem's conditions.

[5]"Properties of the DTFT" <http://cnx.org/content/m0506/latest/>
[6]"Discrete-Time Signals and Systems", Figure 2: Unit sample <http://cnx.org/content/m10342/latest/#fig2>

# Solutions to Exercises in this Module

**Solution to Exercise (p. 1)**

$$\begin{aligned} S\left(e^{i2\pi(f+1)}\right) &= \sum_{n=-\infty}^{\infty} s(n)\, e^{-(i2\pi(f+1)n)} \\ &= \sum_{n=-\infty}^{\infty} e^{-(i2\pi n)} s(n)\, e^{-(i2\pi f n)} \\ &= \sum_{n=-\infty}^{\infty} s(n)\, e^{-(i2\pi f n)} \\ &= S\left(e^{i2\pi f}\right) \end{aligned} \quad (17)$$

**Solution to Exercise (p. 4)**

$$\alpha \sum_{n=n_0}^{N+n_0-1} \alpha^n - \sum_{n=n_0}^{N+n_0-1} \alpha^n = \alpha^{N+n_0} - \alpha^{n_0}$$

which, after manipulation, yields the geometric sum formula.

**Solution to Exercise (p. 6)**

If the sampling frequency exceeds the Nyquist frequency, the spectrum of the samples equals the analog spectrum, but over the normalized analog frequency $fT$. Thus, the energy in the sampled signal equals the original signal's energy multiplied by $T$.