Cauchy's Theorem*

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Abstract

In this module we begin by discussing integrations of complex functions over complex curves and end with Cauchy's Theorem.

1 Introduction

Our main goal is a better understanding of the partial fraction expansion of a given transfer function. With respect to the example that closed the discussion of complex differentiation, see this equation [1] - In this equation[2], we found

$$(zI-B)^{-1} = \frac{1}{z-\lambda_1}P_1 + \frac{1}{(z-\lambda_1)^2}D_1 + \frac{1}{z-\lambda_2}P_2$$

where the P_j and D_j enjoy the amazing properties

1. $$\begin{aligned} BP_1 &= P_1B \\ &= \lambda_1 P_1 + D_1 \end{aligned} \quad (1)$$

   and

   $$BP_2 = P_2B = \lambda_2 P_2$$

2. $$P_1 + P_2 = I \quad (2)$$

   $$P_1^{\,2} = P_1$$

   $$P_2^{\,2} = P_2$$

   and

   $$D_1^{\,2} = 0$$

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[1]"Complex Differentiation", (13) <http://cnx.org/content/m10276/latest/#eq13>
[2]"Complex Differentiation", (17) <http://cnx.org/content/m10276/latest/#eq17>

3.

$$\begin{array}{rcl} P_1 D_1 & = & D_1 P_1 \\ & = & D_1 \end{array} \quad (3)$$

and

$$P_2 D_1 = D_1 P_2 = 0$$

In order to show that this **always** happens, *i.e.*, that it is not a quirk produced by the particular B in this equation[3], we require a few additional tools from the theory of complex variables. In particular, we need the fact that partial fraction expansions may be carried out through complex integration.

2 Integration of Complex Functions Over Complex Curves

We shall be integrating complex functions over complex curves. Such a curve is parameterized by one complex valued or, equivalently, two real valued, function(s) of a real parameter (typically denoted by t). More precisely,

$$C \equiv \{ z(t) = x(t) + iy(t) \mid a \leq t \leq b \}$$

For example, if $x(t) = y(t) = t$ while $a = 0$ and $b = 1$, then C is the line segment joining $0 + i0$ to $1 + i$.

We now define

$$\int f(z)\, dz \equiv \int_a^b f(z(t))\, z'(t)\, dt$$

For example, if $C = \{ t + it \mid 0 \leq t \leq 1 \}$ as above and $f(z) = z$ then

$$\int z dz = \int_0^1 (t + it)(1 + i)\, dt = \int_0^1 t - t + i2t dt = i$$

while if C is the unit circle $\left\{ e^{it} \mid 0 \leq t \leq 2\pi \right\}$ then

$$\int z dz = \int_0^{2\pi} e^{it} i e^{it} dt = i \int_0^{2\pi} e^{i2t} dt = i \int_0^{2\pi} \cos(2t) + i\sin(2t)\, dt = 0$$

Remaining with the unit circle but now integrating $f(z) = \frac{1}{z}$ we find

$$\int z^{-1} dz = \int_0^{2\pi} e^{-(it)} i e^{it} dt = 2\pi i$$

We generalize this calculation to arbitrary (integer) powers over arbitrary circles. More precisely, for integer m and fixed complex a we integrate $(z - a)^m$ over

$$C(a, r) \equiv \left\{ a + re^{it} \mid 0 \leq t \leq 2\pi \right\}$$

the circle of radius r centered at a. We find

$$\begin{array}{rcl} \int (z-a)^m dz & = & \int_0^{2\pi} \left(a + re^{it} - a \right)^m r i e^{it} dt \\ & = & i r^{m+1} \int_0^{2\pi} e^{i(m+1)t} dt \end{array} \quad (4)$$

$$\int (z-a)^m dz = i r^{m+1} \int_0^{2\pi} \cos((m+1)t) + i\sin((m+1)t)\, dt = \begin{cases} 2\pi i & \text{if } m = -1 \\ 0 & \text{otherwise} \end{cases}$$

[3]"Complex Differentiation", (13) <http://cnx.org/content/m10276/latest/#eq13>

When integrating more general functions it is often convenient to express the integral in terms of its real and imaginary parts. More precisely

$$\int f(z)\,dz = \int u(x,y) + iv(x,y)\,dx + i\int u(x,y) + iv(x,y)\,dy$$

$$\int f(z)\,dz = \int u(x,y)\,dx - \int v(x,y)\,dy + i\int v(x,y)\,dx + i\int u(x,y)\,dy$$

$$\int f(z)\,dz = \int_a^b u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t)\,dt + i\int_a^b u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'(t)\,dt$$

The second line should invoke memories of:

Theorem 1: Green's Theorem

If C is a closed curve and M and N are continuously differentiable real-valued functions on C_{in}, the region enclosed by C, then

$$\int M dx + \int N dy = \int\int \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} dxdy$$

Applying this to the situation above, we find, so long as C is closed, that

$$\int f(z)\,dz = -\int\int \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} dxdy + i\int\int \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} dxdy$$

At first glance it appears that Green's Theorem only serves to muddy the waters. Recalling the Cauchy-Riemann equations[4] however we find that each of these double integrals is in fact identically zero! In brief, we have proven:

Theorem 2: Cauchy's Theorem

If f is differentiable on and in the closed curve C then $\int f(z)\,dz = 0$.

Strictly speaking, in order to invoke Green's Theorem we require not only that f be differentiable but that its derivative in fact be continuous. This however is simply a limitation of our simple mode of proof; Cauchy's Theorem is true as stated.

This theorem, together with (4), permits us to integrate every proper rational function. More precisely, if $q = \frac{f}{g}$ where f is a polynomial of degree at most $m-1$ and g is an mth degree polynomial with h distinct zeros at $\{\lambda_j \mid j = \{1,\dots,h\}\}$ with respective multiplicities of $\{m_j \mid j = \{1,\dots,h\}\}$ we found that

$$q(z) = \sum_{j=1}^{h}\sum_{k=1}^{m_j} \frac{q_{j,k}}{(z-\lambda_j)^k} \tag{5}$$

Observe now that if we choose r_j so small that λ_j is the only zero of g encircled by $C_j \equiv C(\lambda_j, r_j)$ then by Cauchy's Theorem

$$\int q(z)\,dz = \sum_{k=1}^{m_j} q_{j,k}\int \frac{1}{(z-\lambda_j)^k} dz$$

In (4) we found that each, save the first, of the integrals under the sum is in fact zero. Hence,

$$\int q(z)\,dz = 2\pi i q_{j,1} \tag{6}$$

[4]"Complex Differentiation": Section Cauchy-Reimann Equations <http://cnx.org/content/m10276/latest/#sec4>

With $q_{j,1}$ in hand, say from this equation[5] or `residue`, one may view (6) as a means for computing the indicated integral. The opposite reading, *i.e.*, that the integral is a convenient means of expressing $q_{j,1}$, will prove just as useful. With that in mind, we note that the remaining residues may be computed as integrals of the product of q and the appropriate factor. More precisely,

$$\int q(z)(z-\lambda_j)^{k-1}dz = 2\pi i q_{j,k} \tag{7}$$

One may be led to believe that the precision of this result is due to the very special choice of curve and function. We shall see ...

[5]"Complex Differentiation", (12) <http://cnx.org/content/m10276/latest/#eq12>