Complex Differentiation*

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Abstract

This module discusses complex differentiation and shows its similarity and differences to real differentiation.

1 Complex Differentiation

The complex f is said to be differentiable at z_0 if

$$\underset{z \to z_0}{\text{limit}} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, by which we mean that

$$\frac{f(z_n) - f(z_0)}{z_n - z_0}$$

converges to the same value **for every** sequence $\{z_n\}$ that converges to z_0. In this case we naturally call the limit $\frac{d}{dz} f(z_0)$

To illustrate the concept of '**for every**' mentioned above, we utilize the following picture. We assume the point z_0 is differentiable, which means that any conceivable sequence is going to converge to z_0. We outline three sequences in the picture: real numbers, imaginary numbers, and a spiral pattern of both.

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Sequences Approaching A Point In The Complex Plane

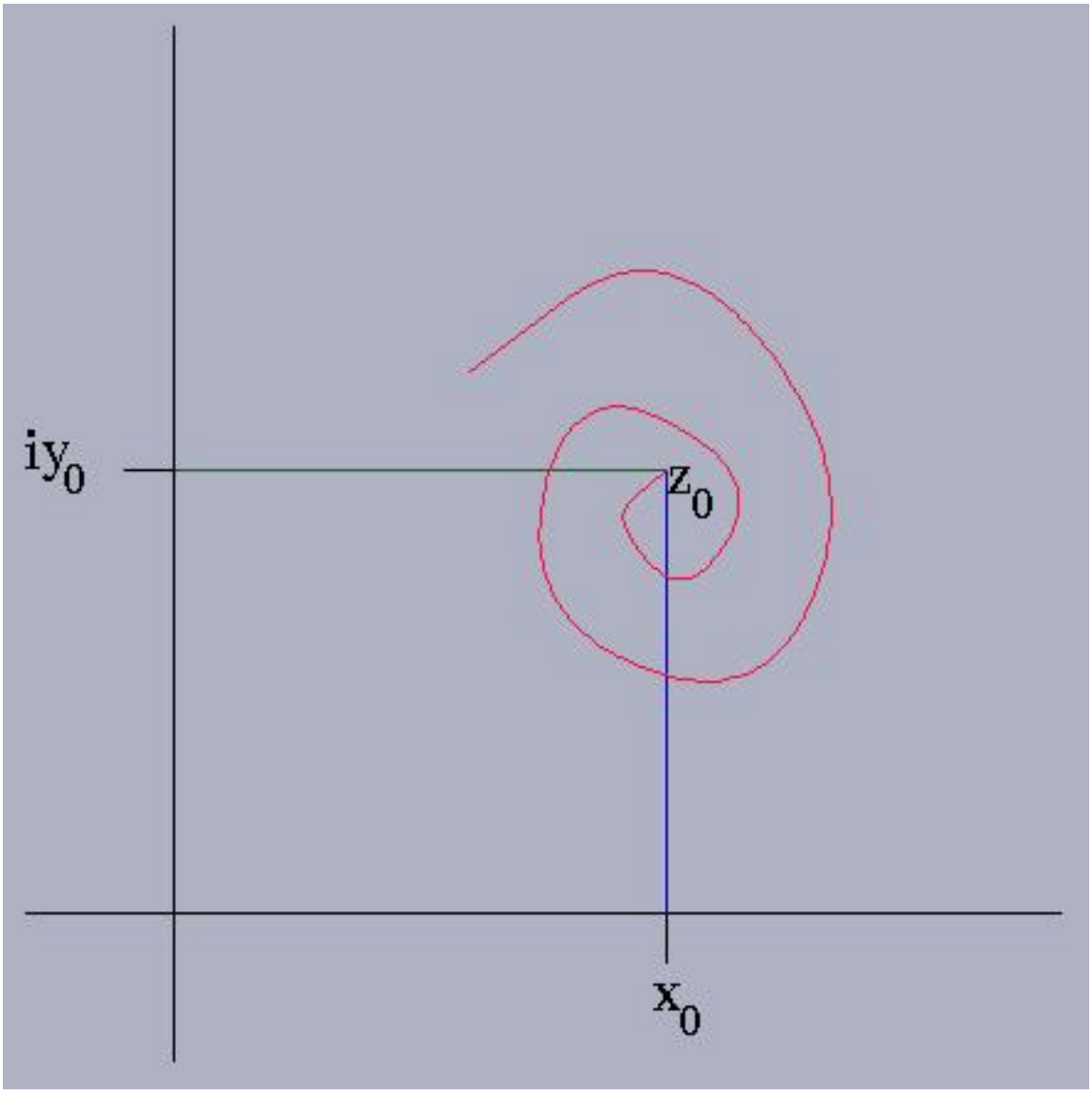

Figure 1: The green is real, the blue is imaginary, and the red is the spiral.

2 Examples

Example 1
The derivative of z^2 is $2z$.

$$\begin{aligned} \underset{z\to z_0}{\text{limit}}\ \frac{z^2-{z_0}^2}{z-z_0} &= \underset{z\to z_0}{\text{limit}}\ \frac{(z-z_0)(z+z_0)}{z-z_0} \\ &= 2z_0 \end{aligned} \tag{1}$$

Example 2
The exponential is its own derivative.

$$\begin{aligned} \underset{z\to z_0}{\text{limit}}\ \frac{e^z-e^{z_0}}{z-z_0} &= e^{z_0}\underset{z\to z_0}{\text{limit}}\ \frac{e^{z-z_0}-1}{z-z_0} \\ &= e^{z_0}\underset{z\to z_0}{\text{limit}}\ \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(n+1)!} \\ &= e^{z_0} \end{aligned} \tag{2}$$

Example 3
The real part of z is **not** a differentiable function of z.

We show that the limit depends on the angle of approach. First, when $z_n \to z_0$ on a line parallel to the real axis, *e.g.*, $z_n = x_0 + \frac{1}{n} + iy_0$, we find

$$\underset{n\to\infty}{\text{limit}}\ \frac{x_0+\frac{1}{n}-x_0}{x_0+\frac{1}{n}+iy_0-x_0+iy_0} = 1 \tag{3}$$

while if $z_n \to z_0$ in the imaginary direction, *e.g.*, $z_n = x_0 + i\left(y_0+\frac{1}{n}\right)$, then

$$\underset{n\to\infty}{\text{limit}}\ \frac{x_0-x_0}{x_0+i\left(y_0+\frac{1}{n}\right)-x_0+iy_0} = 0 \tag{4}$$

3 Conclusion

This last example suggests that when f is differentiable a simple relationship must bind its partial derivatives in x and y.

Proposition 1: Partial Derivative Relationship
If f is differentiable at z_0 then $\frac{d}{dz}f(z_0) = \frac{\partial f(z_0)}{\partial x} = -\left(i\frac{\partial f(z_0)}{\partial y}\right)$
Proof:
With $z = x + iy_0$,

$$\begin{aligned} \frac{d}{dz}f(z_0) &= \underset{z\to z_0}{\text{limit}}\ \frac{f(z)-f(z_0)}{z-z_0} \\ &= \underset{x\to x_0}{\text{limit}}\ \frac{f(x+iy_0)-f(x_0+iy_0)}{x-x_0} \\ &= \frac{\partial f(z_0)}{\partial x} \end{aligned} \tag{5}$$

Alternatively, when $z = x_0 + iy$ then

$$\begin{aligned} \frac{d}{dz}f(z_0) &= \underset{z\to z_0}{\text{limit}}\ \frac{f(z)-f(z_0)}{z-z_0} \\ &= \underset{y\to y_0}{\text{limit}}\ \frac{f(x_0+iy)-f(x_0+iy_0)}{i(y-y_0)} \\ &= -\left(i\frac{\partial f(z_0)}{\partial y}\right) \end{aligned} \tag{6}$$

4 Cauchy-Reimann Equations

In terms of the real and imaginary parts of f this result brings the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{7}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{8}$$

Regarding the converse proposition we note that when f has continuous partial derivatives in region obeying the Cauchy-Reimann equations then f is in fact differentiable in the region.

We remark that with no more energy than that expended on their real cousins one may uncover the rules for differentiating complex sums, products, quotients, and compositions.

As one important application of the derivative let us attempt to expand in partial fractions a rational function whose denominator has a root with degree larger than one. As a warm-up let us try to find $q_{1,1}$ and $q_{1,2}$ in the expression

$$\frac{z+2}{(z+1)^2} = \frac{q_{1,1}}{z+1} + \frac{q_{1,2}}{(z+1)^2}$$

Arguing as above, it seems wise to multiply through by $(z+1)^2$ and so arrive at

$$z+2 = q_{1,1}(z+1) + q_{1,2} \tag{9}$$

On setting $z=-1$ this gives $q_{1,2}=1$. With $q_{1,2}$ computed, (9) takes the simple form $z+1 = q_{1,1}(z+1)$ and so $q_{1,2}=1$ as well. Hence,

$$\frac{z+2}{(z+1)^2} = \frac{1}{z+1}\frac{1}{(z+1)^2}$$

This latter step grows more cumbersome for roots of higher degrees. Let us consider

$$\frac{(z+2)^2}{(z+1)^3} = \frac{q_{1,1}}{z+1} + \frac{q_{1,2}}{(z+1)^2} + \frac{q_{1,3}}{(z+1)^3}$$

The first step is still correct: multiply through by the factor at its highest degree, here 3. This leaves us with

$$(z+2)^2 = q_{1,1}(z+1)^2 + q_{1,2}(z+1) + q_{1,3} \tag{10}$$

Setting $z=-1$ again produces the last coefficient, here $q_{1,3}=1$. We are left however with one equation in two unknowns. Well, not really one equation, for (10) is to hold for **all**z. We exploit this by taking two derivatives, with respect to z, of (10). This produces

$$2(z+2) = 2q_{1,1}(z+1) + q_{1,2}$$

and $2 = q_{1,1}$ The latter of course needs no comment. We derive $q_{1,2}$ from the former by setting $z=-1$. This example will permit us to derive a simple expression for the **partial fraction expansion** of the general proper rational function $q = \frac{f}{g}$ where g has h distinct roots $\{\lambda_1, \ldots, \lambda_h\}$ of respective degrees $\{d_1, \ldots, d_h\}$. We write

$$q(z) = \sum_{j=1}^{h}\sum_{k=1}^{d_j} \frac{q_{j,k}}{(z-\lambda_j)^k} \tag{11}$$

and note, as above, that $q_{j,k}$ is the coefficient of $(z-d_j)^{d_j-k}$ in the rational function

$$r_j(z) \equiv q(z)(z-\lambda_j)^{d_j}$$

Hence, $q_{j,k}$ may be computed by setting $z=\lambda_j$ in the ratio of the d_j-kth derivative of r_j to $(d_j-k)!$. That is,

$$q_{j,k} = \lim_{z\to\lambda_j} \frac{1}{(d_j-k)!}\frac{d^{d_j-k}}{dz^{d_j-k}}\left\{(z-\lambda_j)^{d_j} q(z)\right\} \tag{12}$$

As a second example, let us take

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{13}$$

and compute the $\Phi_{j,k}$ matrices in the expansion

$$(zI-B)^{-1} = \begin{pmatrix} \frac{1}{z-1} & 0 & 0 \\ \frac{1}{(z-1)(z-3)} & \frac{1}{z-3} & 0 \\ \frac{1}{(z-1)^2(z-3)} & \frac{1}{(z-1)(z-3)} & \frac{1}{z-1} \end{pmatrix} = \frac{1}{z-1}\Phi_{1,1} + \frac{1}{(z-1)^2}\Phi_{1,2} + \frac{1}{z-3}\Phi_{2,1}$$

The only challenging term is the $(3,1)$ element. We write

$$\frac{1}{(z-1)^2(z-3)} = \frac{q_{1,1}}{z-1} + \frac{q_{1,2}}{(z-1)^2} + \frac{q_{2,1}}{z-3}$$

It follows from (12) that

$$\begin{aligned} q_{1,1} &= \tfrac{d}{dz}\left(\tfrac{1}{z-3}1\right) \\ &= -1/4 \end{aligned} \tag{14}$$

and

$$\begin{aligned} q_{1,2} &= \tfrac{1}{z-3}1 \\ &= -1/4 \end{aligned} \tag{15}$$

and

$$\begin{aligned} q_{2,1} &= \left(\tfrac{1}{z-3}\right)^2 3 \\ &= 1/4 \end{aligned} \tag{16}$$

It now follows that

$$(zI-B)^{-1} = \frac{1}{z-1}\begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 0 & 0 \\ -1/4 & -1/2 & 1 \end{pmatrix} + \frac{1}{(z-1)^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix} + \frac{1}{z-3}\begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & 1/2 & 0 \end{pmatrix} \tag{17}$$

In closing, let us remark that the method of partial fraction expansions has been implemented in Matlab. In fact, (14), (15), and (16) all follow from the single command: `[r,p,k]=residue([0 0 0 1],[1 -5 7 -3])`. The first input argument is Matlab-speak for the polynomial $f(z)=1$ while the second argument corresponds to the denominator

$$g(z) = (z-1)^2(z-3) = z^3 - 5z^2 + 7z - 3$$

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