Vector Space*

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Abstract

This module discusses vector spaces and their applications to complex arithmetic.

1 Introduction

You have long taken for granted the fact that the set of real numbers, $\mathbb{R}$, is closed under addition and multiplication, that each number has a unique additive inverse, and that the commutative, associative, and distributive laws were right as rain. The set, $\mathbb{C}$, of complex numbers also enjoys each of these properties, as do the sets $\mathbb{R}^n$ and $\mathbb{C}^n$ of columns of n real and complex numbers, respectively.

To be more precise, we write $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ as

$\mathbf{x} = (x_1, x_2, \dots, x_n)^T$

$\mathbf{y} = (y_1, y_2, \dots, y_n)^T$

and define their vector sum as the elementwise sum

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \quad (1)$$

and similarly, the product of a complex scalar, $\mathbf{z} \in \mathbb{C}$ with $\mathbf{x}$ as:

$$\mathbf{z}\mathbf{x} = \begin{pmatrix} \mathbf{z}x_1 \\ \mathbf{z}x_2 \\ \vdots \\ \mathbf{z}x_n \end{pmatrix} \quad (2)$$

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2 Vector Space

These notions lead naturally to the concept of vector space. A set V is said to be a vector space if

1. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for each $\mathbf{x}$ and $\mathbf{y}$ in V
2. $\mathbf{x}+\mathbf{y}+\mathbf{z}=\mathbf{x}+\mathbf{y}+\mathbf{z}$ for each $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in V
3. There is a unique "zero vector" such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for each $\mathbf{x}$ in V
4. For each $\mathbf{x}$ in V there is a unique vector $-\mathbf{x}$ such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$.
5. $1\mathbf{x}=\mathbf{x}$
6. $(c_1c_2)\,\mathbf{x}=c_1\,(c_2\mathbf{x})$ for each $\mathbf{x}$ in V and c_1 and c_2 in $\mathbb{C}$.
7. $c\,(\mathbf{x}+\mathbf{y})=c\mathbf{x}+c\mathbf{y}$ for each $\mathbf{x}$ and $\mathbf{y}$ in V and c in $\mathbb{C}$.
8. $(c_1+c_2)\,\mathbf{x}=c_1\mathbf{x}+c_2\mathbf{x}$ for each $\mathbf{x}$ in V and c_1 and c_2 in $\mathbb{C}$.