Uniform Sampling of a Continuous-Time Bandlimited Signal*

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Abstract

Covers methods involved in uniform sampling of a continuous-time bandlimited signal.

1 Uniform Sampling

Say that $x[n]$ is sampled from $x_c(t)$ with uniform sampling interval T:

$$\forall n, n \in \mathbb{Z} : (x[n] = x_c(nT)) \tag{1}$$

Let us define the continuous-time "impulse train"

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{2}$$

where $\delta(t)$ denotes the Dirac delta function, defined by the properties:

unit area

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3}$$

sifting property

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - \tau)\, dt = f(\tau) \tag{4}$$

Using Fourier series, it can be shown that

$$p(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{i\frac{2\pi}{T}kt} \tag{5}$$

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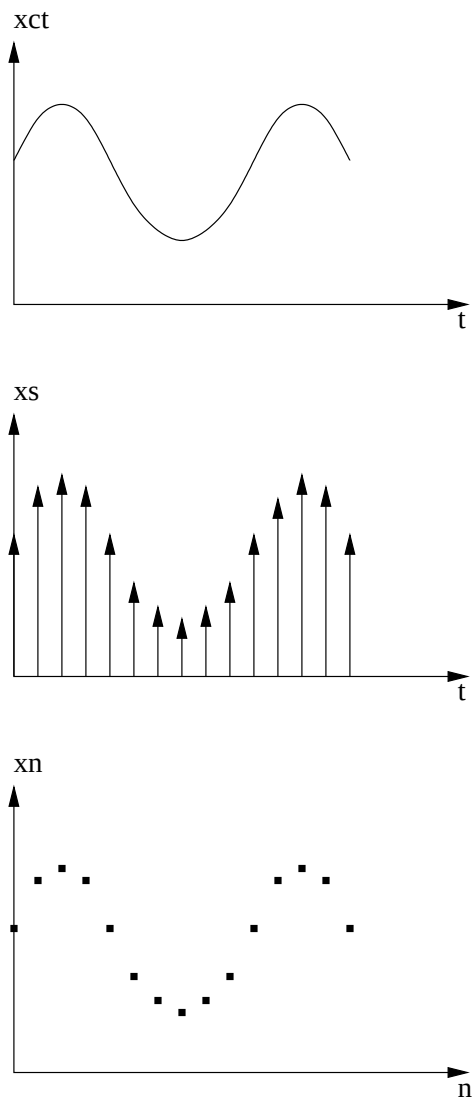

Figure 1: Signals used in sampling theorem.

Multiplying $x_c(t)$ by the impulse train yields the "continuous-time sampled signal" $x_s(t)$ which will help us to derive the sampling theorem. (See Figure 1)

$$\begin{aligned} x_s(t) &= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= x_c(t) \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{i \frac{2\pi}{T} kt} \end{aligned} \tag{6}$$

Taking the CTFT of $x_s(t)$,

$$\begin{aligned} X_s(i\Omega) &= \int_{-\infty}^{\infty}\left(x_c(t)\,\frac{1}{T}\sum_{k=-\infty}^{\infty} e^{i\frac{2\pi}{T}kt}\right)e^{-(i\Omega t)}dt \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} x_c(t)\,e^{-\left(i\left(\Omega-\frac{2\pi}{T}k\right)t\right)}dt \end{aligned} \tag{7}$$

$$X_s(i\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(i\left(\Omega-\frac{2\pi}{T}k\right)\right) \tag{8}$$

Notice that $X_s(i\Omega)$ is a summation of scaled and shifted copies of $X_c(i\Omega)$. When $X_c(i\Omega)$ is bandlimited to $\frac{\pi}{T}\frac{\text{rad}}{s}$ the spectral copies do not overlap, and we say there is no "aliasing." Aliasing may result when $X_c(i\Omega)$ is not bandlimited to $\frac{\pi}{T}$. (See Figure 2 and Figure 3) The frequency $\frac{\pi}{T}\frac{\text{rad}}{s}$, *i.e.*, $\frac{1}{2T}$Hz, is often called the **Nyquist frequency**.

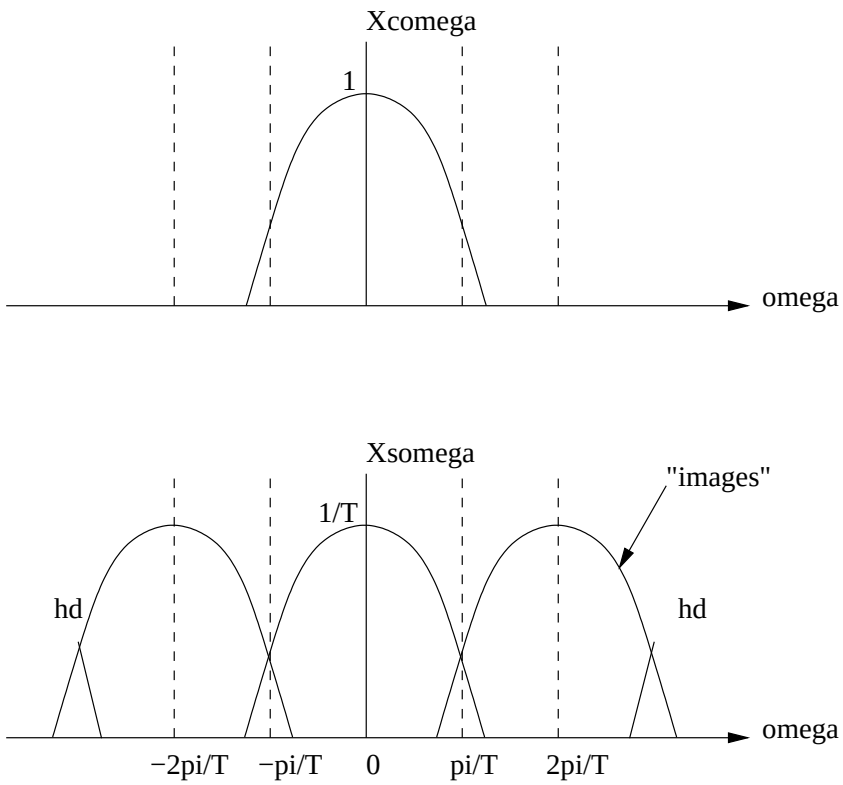

Figure 2: Example of aliasing in $X_s(i\Omega)$

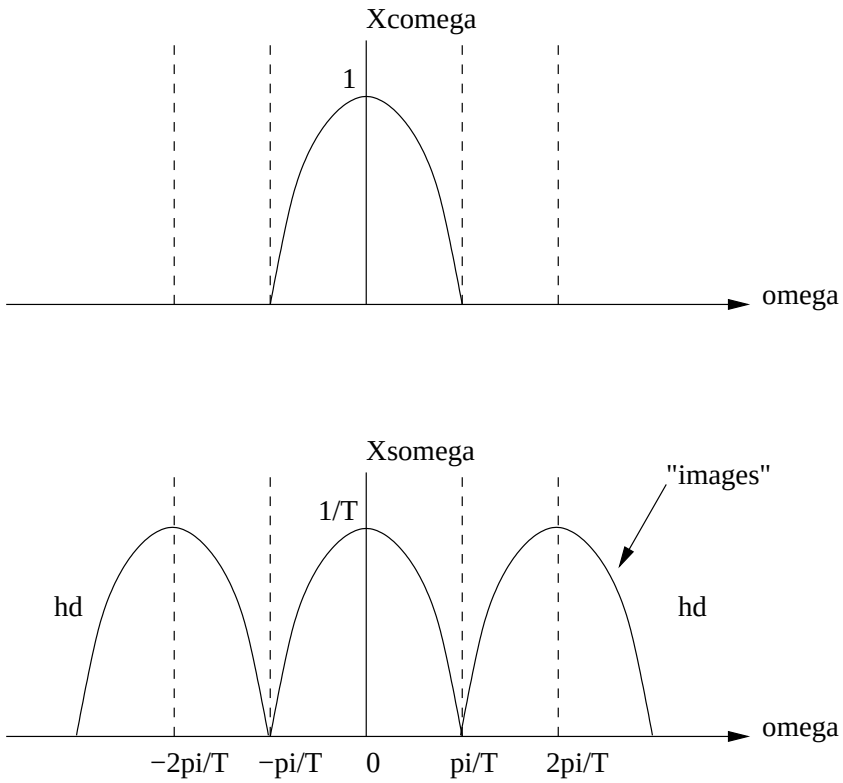

Figure 3: Example of no aliasing in $X_s(i\Omega)$

We now investigate the relationship between the CTFT and the DTFT:

$$\begin{aligned} X\left(e^{i^{\omega}}\right) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-(i\omega n)} \\ &= \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-(i\omega n)} \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x_c(t)\, \delta(t-nT)\, dt e^{-(i\omega n)} \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x_c(t)\, \delta(t-nT)\, e^{-(i\omega n)} dt \end{aligned} \tag{9}$$

where $x_c(t)\,\delta(t-nT)\,e^{-(i\omega n)}$ is nonzero $\Leftrightarrow n = \frac{t}{T}$

$$\begin{aligned} X\left(e^{i^{\omega}}\right) &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x_c(t)\, \delta(t-nT)\, e^{-\left(i\omega \frac{t}{T}\right)} dt \\ &= \int_{-\infty}^{\infty} x_c(t) \sum_{n=-\infty}^{\infty} e^{-\left(i\frac{\omega}{T}t\right)} dt \end{aligned} \tag{10}$$

where

$$x_c(t) \sum_{n=-\infty}^{\infty} \delta(t-nT) = x_s(t)$$

$$X\left(e^{i^{\omega}}\right) = X_s\left(i\frac{\omega}{T}\right) \tag{11}$$

Plugging (8) into (11) yields:

$$X\left(e^{i^{\omega}}\right)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(i\frac{\omega-2\pi k}{T}\right) \tag{12}$$

Notice that the DTFT of the sampled signal $x[n]$ is a summation of scaled, stretched, and shifted copies of the CTFT of the continuous signal $x_c(t)$. As implied by (11), the DTFT may show evidence of aliasing when $x_c(t)$ is not bandlimited to the Nyquist frequency. (See Figure 4 and Figure 5)

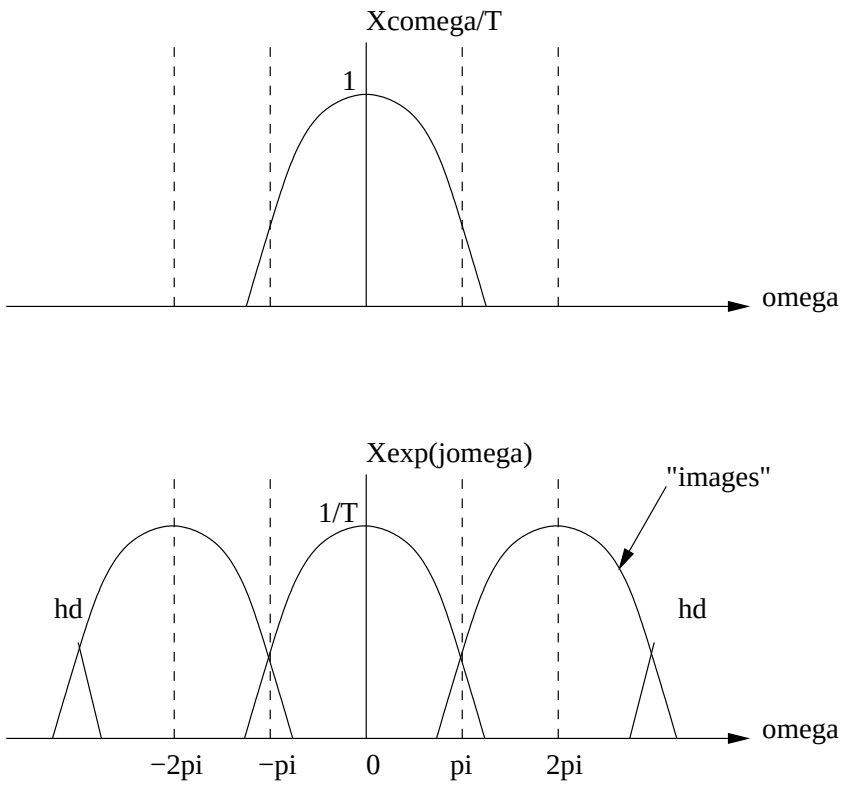

Figure 4: Example of aliasing in $X\left(e^{i\omega}\right)$.

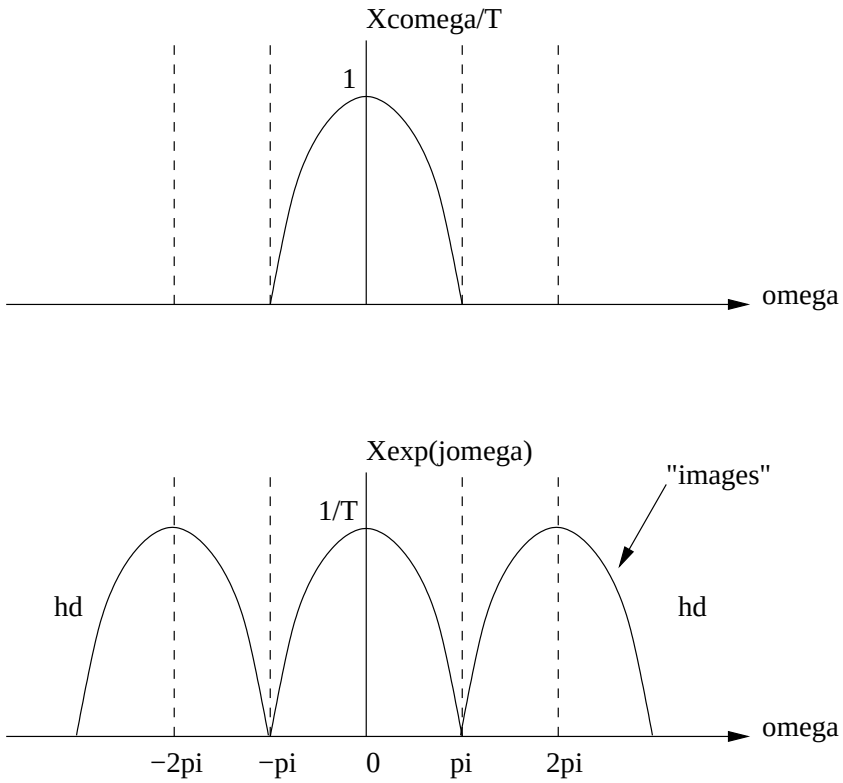

Figure 5: Example of no aliasing in $X\left(e^{i\omega}\right)$.