Continuous Wavelet Transform*

Phil Schniter

Abstract

This module introduces continuous wavelet transform.

The STFT provided a means of (joint) time-frequency analysis with the property that spectral/temporal widths (or resolutions) were the same for all basis elements. Let's now take a closer look at the implications of uniform resolution.

Consider two signals composed of sinusoids with frequency 1 Hz and 1.001 Hz, respectively. It may be difficult to distinguish between these two signals in the presence of background noise unless many cycles are observed, implying the need for a many-second observation. Now consider two signals with pure frequencies of 1000 Hz and 1001 Hz-again, a 0.1% difference. Here it should be possible to distinguish the two signals in an interval of much less than one second. In other words, good frequency resolution requires longer observation times as frequency decreases. Thus, it might be more convenient to construct a basis whose elements have larger temporal width at low frequencies.

The previous example motivates a multi-resolution time-frequency tiling of the form (Figure 1):

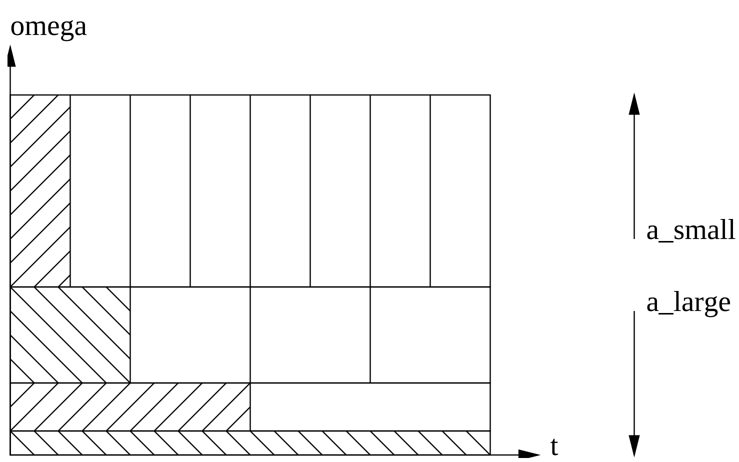

Figure 1

*Version 2.14: May 31, 2009 6:11 pm -0500
†http://creativecommons.org/licenses/by/1.0

The Continuous Wavelet Transform (CWT) accomplishes the above multi-resolution tiling by time-scaling and time-shifting a prototype function $\psi(t)$, often called the **mother wavelet**. The a-scaled and τ-shifted basis elements is given by

$$\psi_{a,\tau}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-\tau}{a}\right)$$

where

$$a \wedge \tau \in \mathbb{R}$$

$$\int_{-\infty}^{\infty}\psi(t)\,dt = 0$$

$$C_\psi = \int_{-\infty}^{\infty}\frac{(|\psi(\Omega)|)^2}{|\Omega|}d\Omega < \infty$$

The conditions above imply that $\psi(t)$ is bandpass and sufficiently smooth. Assuming that $\parallel \psi(t) \parallel = 1$, the definition above ensures that $\parallel \psi_{a,\tau}(t) \parallel = 1$ for all a and τ. The CWT is then defined by the transform pair

$$X_{\mathrm{CWT}}(a,\tau) = \int_{-\infty}^{\infty} x(t)\,\overline{\psi_{a,\tau}(t)}dt$$

$$x(t) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{X_{\mathrm{CWT}}(a,\tau)\,\psi_{a,\tau}(t)}{a^2}d\tau da$$

In basis terms, the CWT says that a waveform can be decomposed into a collection of shifted and stretched versions of the mother wavelet $\psi(t)$. As such, it is usually said that wavelets perform a "time-scale" analysis rather than a time-frequency analysis.

The **Morlet wavelet** is a classic example of the CWT. It employs a windowed complex exponential as the mother wavelet:

$$\psi(t) = \frac{1}{\sqrt{2\pi}}e^{-(i\Omega_0 t)}e^{-\frac{t^2}{2}}$$

$$\Psi(\Omega) = e^{-\frac{(\Omega-\Omega_0)^2}{2}}$$

where it is typical to select $\Omega_0 = \pi\sqrt{\frac{2}{\log 2}}$. (See illustration (Figure 2).) While this wavelet does not exactly satisfy the conditions established earlier, since $\Psi(0) \simeq 7\times10^{-7} \neq 0$, it can be corrected, though in practice the correction is negligible and usually ignored.

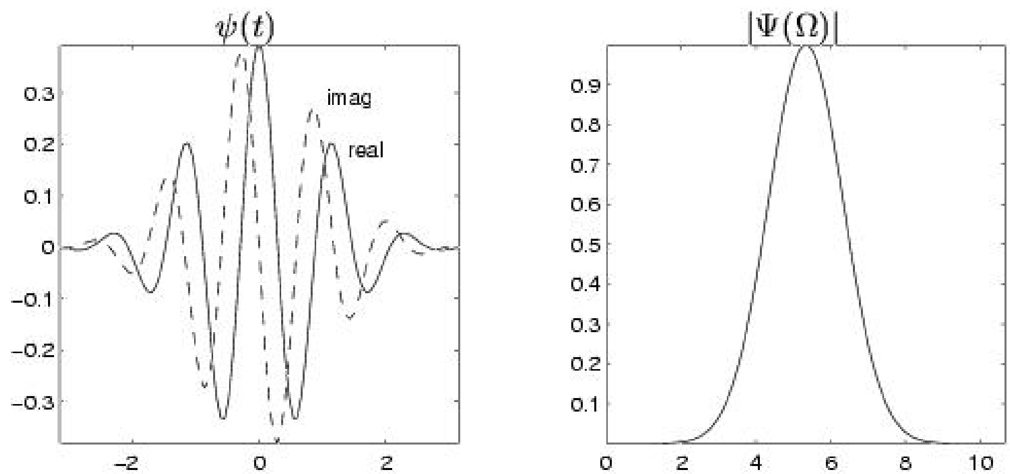

Figure 2

While the CWT discussed above is an interesting theoretical and pedagogical tool, the discrete wavelet transform (DWT) is much more practical. Before shifting our focus to the DWT, we take a step back and review some of the basic concepts from the branch of mathematics known as Hilbert Space theory (Vector Space[1], Normed Vector Space[2], Inner Product Space[3], Hilbert Space[4], Projection Theorem[5]). These concepts will be essential in our development of the DWT.

[1]"Vector Space" <http://cnx.org/content/m10419/latest/>
[2]"Normed Vector Space" <http://cnx.org/content/m10428/latest/>
[3]"Inner Product Space" <http://cnx.org/content/m10430/latest/>
[4]"Hilbert Spaces" <http://cnx.org/content/m10434/latest/>
[5]"Projection Theorem" <http://cnx.org/content/m10435/latest/>