Vector Space*

Phil Schniter

This work is produced by OpenStax-CNX and licensed under the
Creative Commons Attribution License 1.0†

Abstract

This module introduces vector space.

- A **vector space** consists of the following four elements:
  a. A set of vectors V,
  b. A field of scalars $\mathbb{F}$ (where, for our purposes, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$),
  c. The operations of vector addition "+" (*i.e.*, $+ : V \times V \rightarrow V$)
  d. The operation of scalar multiplication "·"(*i.e.*, $\cdot : \mathbb{F} \times V \rightarrow V$)

  for which the following properties hold. (Assume $x \ \wedge \ y \ \wedge \ z \in V$ and $\alpha \ \wedge \ \beta \in \mathbb{F}$.)

Properties	Examples
commutativity	$x + y = y + x$
associativity	$(x + y) + z = x + (y + z)$
	$(\alpha\beta) x = \alpha (\beta x)$
distributivity	$\alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y)$
	$(\alpha + \beta) x = \alpha x + \beta x$
additive identity	$\forall x \in V : (\exists 0, 0 \in V : (x + 0 = x))$
additive inverse	$\forall x \in V : (\exists -\mathrm{x}, (-x) \in V : (x + -x = 0))$
multiplicative identity	$\forall x \in V : (1 \cdot x = x)$

Table 1

Important examples of vector spaces include

Properties	Examples
real N-vectors	$V = \mathbb{R}^N$, $\mathbb{F} = \mathbb{R}$
complex N-vectors	$V = \mathbb{C}^N$, $\mathbb{F} = \mathbb{C}$
sequences in "l_p"	$V = \left\{ x[n] \mid \exists n \in \mathbb{Z} : \left(\sum_{n=-\infty}^{\infty} (\lvert x[n] \rvert)^p < \infty \right) \right\}$, $\mathbb{F} = \mathbb{C}$
functions in "$\mathcal{L}_p$"	$V = \left\{ f(t) \mid \int_{-\infty}^{\infty} (\lvert f(t) \rvert)^p dt < \infty \right\}$, $\mathbb{F} = \mathbb{C}$

*Version 2.13: Oct 18, 2005 4:34 pm -0500
†http://creativecommons.org/licenses/by/1.0

Table 2

where we have assumed the usual definitions of addition and multiplication. From now on, we will denote the arbitrary vector space $(V, \mathbb{F}, +, \cdot)$ by the shorthand V and assume the usual selection of $(\mathbb{F}, +, \cdot)$. We will also suppress the "$\cdot$" in scalar multiplication, so that $\alpha \cdot x$ becomes αx.

- A **subspace** of V is a subset $M \subset V$ for which
  a. $\forall x, y, x \in M \ \wedge \ y \in M : ((x + y) \in M)$
  b. $\forall x \in M \ \wedge \ \alpha \in \mathbb{F} : (\alpha x \in M)$

  NOTE: Note that every subspace must contain 0, and that V is a subspace of itself.

- The **span** of set $S \subset V$ is the subspace of V containing all linear combinations of vectors in S. When $S = \{x_0, \ldots, x_{N-1}\}$,

$$\text{span}\,(S) := \left\{ \sum_{i=0}^{N-1} \alpha_i x_i \mid \alpha_i \in \mathbb{F} \right\}$$

- A subset of linearly-independent vectors $\{x_0, \ldots, x_{N-1}\} \subset V$ is called a **basis for V** when its span equals V. In such a case, we say that V has **dimension**N. We say that V is **infinite-dimensional**[1] if it contains an infinite number of linearly independent vectors.
- V is a **direct sum** of two subspaces M and N, written $V = M \oplus N$, iff every $x \in V$ has a **unique** representation $x = m + n$ for $m \in M$ and $n \in N$.

  NOTE: Note that this requires $M \cap N = \{0\}$

[1] The definition of an infinite-dimensional basis would be complicated by issues relating to the convergence of infinite series. Hence we postpone discussion of infinite-dimensional bases until the Hilbert Space (<http://cnx.org/content/m10434/latest/>) section.