Discrete Fourier Transformation*

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Abstract

This module covers the fundamentals of Discrete-Fourier Transformations.

1 N-point Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} \left(x[n] e^{(-i)\frac{2\pi}{n}kn} \right) \forall k, k = \{0, \dots, N-1\} \tag{1}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \left(X[k] e^{i\frac{2\pi}{n}kn} \right) \forall n, n = \{0, \dots, N-1\} \tag{2}$$

Note that:

- $X[k]$ is the DTFT evaluated at $\omega = \frac{2\pi}{N} k \forall k, k = \{0, \dots, N-1\}$
- Zero-padding $x[n]$ to M samples prior to the DFT yields an M-point uniform sampled version of the DTFT:

$$X\left(e^{i\frac{2\pi}{M}k}\right) = \sum_{n=0}^{N-1} \left(x[n] e^{(-i)\frac{2\pi}{M}k} \right) \tag{3}$$

$$X\left(e^{i\frac{2\pi}{M}k}\right) = \sum_{n=0}^{N-1} \left(x_{\text{zp}}[n] e^{(-i)\frac{2\pi}{M}k} \right)$$

$$X\left(e^{i\frac{2\pi}{M}k}\right) = X_{\text{zp}}[k] \forall k, k = \{0, \dots, M-1\}$$

- The N-pt DFT is sufficient to reconstruct the entire DTFT of an N-pt sequence:

$$X\left(e^{i\omega}\right) = \sum_{n=0}^{N-1} \left(x[n] e^{(-i)\omega n} \right) \tag{4}$$

$$X\left(e^{i\omega}\right) = \left(\sum_{n=0}^{N-1} \left(\frac{1}{N} \right) \right) \sum_{k=0}^{N-1} \left(X[k] e^{i\frac{2\pi}{N}kn} e^{(-i)\omega n} \right)$$

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$$X\left(e^{i\omega}\right) = \left(\sum_{k=0}^{N-1}\left(X[k]\right)\right)\frac{1}{N}\sum_{k=0}^{N-1}\left(e^{(-i)\left(\omega-\frac{2\pi}{N}k\right)n}\right)$$

$$X\left(e^{i\omega}\right) = \left(\sum_{k=0}^{N-1}\left(X[k]\right)\right)\frac{1}{N}\left(\frac{\sin\left(\frac{\omega N-2\pi k}{2}\right)}{\sin\left(\frac{\omega N-2\pi k}{2N}\right)}e^{(-i)\left(\omega-\frac{2\pi}{N}k\right)\frac{N-1}{2}}\right)$$

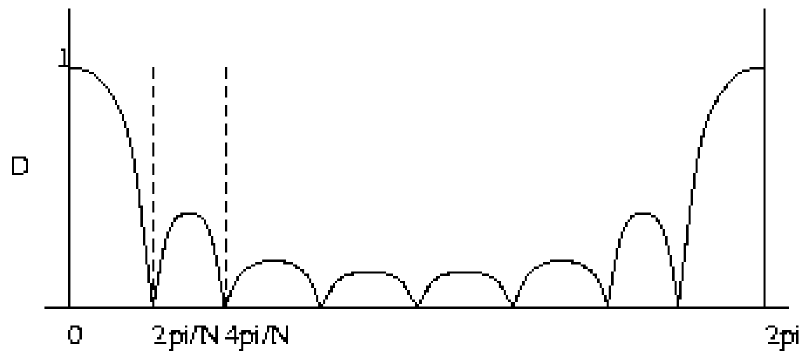

Figure 1: Dirichlet sinc, $\frac{1}{N}\frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$

- The DFT has a convenient matrix representation. Defining $W_N = e^{(-i)\frac{2\pi}{N}}$,

$$\begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 & \dots \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 & \dots \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix} \quad (5)$$

  where $X = W(x)$ respectively. W has the following properties:
  - W is Vandermonde: the nth column of W is a polynomial in W_N^n
  - W is symmetric: $W = W^T$
  - $\frac{1}{\sqrt{N}}W$ is unitary: $\left(\frac{1}{\sqrt{N}}W\right)\left(\frac{1}{\sqrt{N}}W\right)^H = \left(\frac{1}{\sqrt{N}}W\right)^H\left(\frac{1}{\sqrt{N}}W\right) = I$
  - $\frac{1}{N}\overline{W} = W^{-1}$, the IDFT matrix.
- For N a power of 2, the FFT can be used to compute the DFT using about $\frac{N}{2}\log_2 N$ rather than N^2 operations.

N	$\frac{N}{2}\log_2 N$	N^2
16	32	256
64	192	4096
256	1024	65536
1024	5120	1048576

Table 1