Normed Vector Space*

Phil Schniter

Abstract

This module introduces normed vector space.

Now we equip a vector space V with a notion of "size".

- A **norm** is a function ($\parallel \cdot \parallel: V \to \mathbb{R}$) such that the following properties hold ($\forall x, y, x \in V \wedge y \in V : (x \in V \wedge y \in V)$ and $\forall \alpha, \alpha \in \mathbb{F} : (\alpha \in \mathbb{F})$):
    a. $\parallel x \parallel \geq 0$ with equality iff $x = 0$
    b. $\parallel \alpha x \parallel = |\alpha| \cdot \parallel x \parallel$
    c. $\parallel x + y \parallel \leq \parallel x \parallel + \parallel y \parallel$, (the **triangle inequality**).

  In simple terms, the norm measures the size of a vector. Adding the norm operation to a vector space yields a **normed vector space**. Important example include:
    a. $V = \mathbb{R}^N$, $\parallel (x_0, \dots, x_{N-1})^T \parallel := \sqrt{\sum_{i=0}^{N-1} {x_i}^2} := \sqrt{x^T x}$
    b. $V = \mathbb{C}^N$, $\parallel (x_0, \dots, x_{N-1})^T \parallel := \sqrt{\sum_{i=0}^{N-1} (|x_i|)^2} := \sqrt{x^H x}$
    c. $V = l_p$, $\parallel \{x[n]\} \parallel := \left(\sum_{n=-\infty}^{\infty} (|x[n]|)^p\right)^{\frac{1}{p}}$
    d. $V = \mathcal{L}_p$, $\parallel f(t) \parallel := \left(\int_{-\infty}^{\infty} (|f(t)|)^p dt\right)^{\frac{1}{p}}$

*Version 2.14: Oct 4, 2005 10:11 am +0000