Discrete Wavelet Transform: Main Concepts*

Phil Schniter

1 Main Concepts

The **discrete wavelet transform** (DWT) is a representation of a signal $x(t) \in \mathcal{L}_2$ using an orthonormal basis consisting of a countably-infinite set of **wavelets**. Denoting the wavelet basis as $\{\psi_{k,n}(t) \mid k \in \mathbb{Z} \wedge n \in \mathbb{Z}\}$, the DWT transform pair is

$$x(t) = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_{k,n} \psi_{k,n}(t) \qquad (1)$$

$$\begin{array}{rcl} d_{k,n} & = & < \psi_{k,n}(t), x(t) > \\ & = & \int_{-\infty}^{\infty} \overline{\psi_{k,n}(t)} x(t)\, dt \end{array} \qquad (2)$$

where $\{d_{k,n}\}$ are the wavelet coefficients. Note the relationship to Fourier series and to the sampling theorem: in both cases we can perfectly describe a continuous-time signal $x(t)$ using a countably-infinite (*i.e.*, discrete) set of coefficients. Specifically, Fourier series enabled us to describe **periodic** signals using Fourier coefficients $\{X[k] \mid k \in \mathbb{Z}\}$, while the sampling theorem enabled us to describe **bandlimited** signals using signal samples $\{x[n] \mid n \in \mathbb{Z}\}$. In both cases, signals within a limited class are represented using a coefficient set with a single countable index. The DWT can describe **any** signal in $\mathcal{L}_2$ using a coefficient set parameterized by two countable indices: $\{d_{k,n} \mid k \in \mathbb{Z} \wedge n \in \mathbb{Z}\}$.

Wavelets are orthonormal functions in $\mathcal{L}_2$ obtained by shifting and stretching a **mother wavelet**, $\psi(t) \in \mathcal{L}_2$. For example,

$$\forall k, n, k \wedge n \in \mathbb{Z} : \left(\psi_{k,n}(t) = 2^{-\frac{k}{2}} \psi\left(2^{-k} t - n\right)\right) \qquad (3)$$

defines a family of wavelets $\{\psi_{k,n}(t) \mid k \in \mathbb{Z} \wedge n \in \mathbb{Z}\}$ related by power-of-two stretches. As k increases, the wavelet stretches by a factor of two; as n increases, the wavelet shifts right.

NOTE: When $\| \psi(t) \| = 1$, the normalization ensures that $\| \psi_{k,n}(t) \| = 1$ for all $k \in \mathbb{Z}$, $n \in \mathbb{Z}$.

Power-of-two stretching is a convenient, though somewhat arbitrary, choice. In our treatment of the discrete wavelet transform, however, we will focus on this choice. Even with power-of two stretches, there are various possibilities for $\psi(t)$, each giving a different flavor of DWT.

Wavelets are constructed so that $\{\psi_{k,n}(t) \mid n \in \mathbb{Z}\}$ (*i.e.*, the set of all shifted wavelets at fixed scale k), describes a particular level of 'detail' in the signal. As k becomes smaller (*i.e.*, closer to $-\infty$), the wavelets

*Version 2.12: Sep 20, 2005 8:56 pm +0000
†http://creativecommons.org/licenses/by/1.0

become more "fine grained" and the level of detail increases. In this way, the DWT can give a **multi-resolution** description of a signal, very useful in analyzing "real-world" signals. Essentially, the DWT gives us a **discrete multi-resolution description of a continuous-time signal in**$\mathcal{L}_2$.

In the modules that follow, these DWT concepts will be developed "from scratch" using Hilbert space principles. To aid the development, we make use of the so-called **scaling function**$\phi(t) \in \mathcal{L}_2$, which will be used to approximate the signal **up to a particular level of detail**. Like with wavelets, a family of scaling functions can be constructed via shifts and power-of-two stretches

$$\forall k, n, k \wedge n \in \mathbb{Z} : \left(\phi_{k,n}(t) = 2^{-\frac{k}{2}} \phi\left(2^{-k} t - n\right)\right) \tag{4}$$

given mother scaling function $\phi(t)$. The relationships between wavelets and scaling functions will be elaborated upon later via theory[1] and example[2].

> NOTE: The inner-product expression for $d_{k,n}$, (2) is written for the general complex-valued case. In our treatment of the discrete wavelet transform, however, we will assume real-valued signals and wavelets. For this reason, we omit the complex conjugations in the remainder of our DWT discussions

[1]"The Scaling Equation" <http://cnx.org/content/m10476/latest/>
[2]"The Haar System as an Example of DWT" <http://cnx.org/content/m10437/latest/>