

FOURIER SERIES: EIGENFUNCTION APPROACH*

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Abstract

This module will introduce the Fourier Series and its Fourier coefficients using the concepts of eigenfunctions and basis. We will show several examples of how to decompose a signal and find the Fourier coefficients.

1 Introduction

Since complex exponentials¹ are eigenfunctions of linear time-invariant (LTI) systems², calculating the output of an LTI system \mathcal{H} given e^{st} as an input amounts to simple multiplication, where $H(s) \in \mathbb{C}$ is a constant (that depends on s). In the figure (Figure 1) below we have a simple exponential input that yields the following output:

$$y(t) = H(s) e^{st} \quad (1)$$

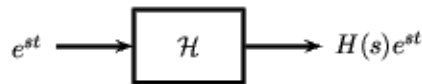


Figure 1: Simple LTI system.

Using this and the fact that \mathcal{H} is linear, calculating $y(t)$ for combinations of complex exponentials is also straightforward. This linearity property is depicted in the two equations below - showing the input to the linear system H on the left side and the output, $y(t)$, on the right:

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¹"The Complex Exponential" <<http://cnx.org/content/m10060/latest/>>

²"Eigenfunctions of LTI Systems" <<http://cnx.org/content/m10500/latest/>>

1.

$$c_1 e^{s_1 t} + c_2 e^{s_2 t} \rightarrow c_1 H(s_1) e^{s_1 t} + c_2 H(s_2) e^{s_2 t}$$

2.

$$\sum_n (c_n e^{s_n t}) \rightarrow \sum_n (c_n H(s_n) e^{s_n t})$$

The action of H on an input such as those in the two equations above is easy to explain: \mathcal{H} **independently scales** each exponential component $e^{s_n t}$ by a different complex number $H(s_n) \in \mathbb{C}$. As such, if we can write a function $f(t)$ as a combination of complex exponentials it allows us to:

- easily calculate the output of \mathcal{H} given $f(t)$ as an input (provided we know the eigenvalues $H(s)$)
- interpret how \mathcal{H} manipulates $f(t)$

2 Fourier Series

Joseph Fourier³ demonstrated that an arbitrary T -periodic function⁴ $f(t)$ can be written as a linear combination of harmonic complex sinusoids

$$f(t) = \sum_{n=-\infty}^{\infty} (c_n e^{i\omega_0 n t}) \quad (2)$$

where $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency. For almost all $f(t)$ of practical interest, there exists c_n to make (2) true. If $f(t)$ is finite energy ($f(t) \in L^2[0, T]$), then the equality in (2) holds in the sense of energy convergence; if $f(t)$ is continuous, then (2) holds pointwise. Also, if $f(t)$ meets some mild conditions (the Dirichlet conditions), then (2) holds pointwise everywhere except at points of discontinuity.

The c_n - called the **Fourier coefficients** - tell us "how much" of the sinusoid $e^{i\omega_0 n t}$ is in $f(t)$. (2) essentially breaks down $f(t)$ into pieces, each of which is easily processed by an LTI system (since it is an eigenfunction of **every** LTI system). Mathematically, (2) tells us that the set of harmonic complex exponentials $\{\forall n, n \in \mathbb{Z} : (e^{i\omega_0 n t})\}$ form a basis for the space of T -periodic continuous time functions. Below are a few examples that are intended to help you think about a given signal or function, $f(t)$, in terms of its exponential basis functions.

2.1 Examples

For each of the given functions below, break it down into its "simpler" parts and find its fourier coefficients. Click to see the solution.

Exercise 1

(Solution on p. 4.)

$$f(t) = \cos(\omega_0 t)$$

Exercise 2

(Solution on p. 4.)

$$f(t) = \sin(2\omega_0 t)$$

Exercise 3

(Solution on p. 4.)

$$f(t) = 3 + 4\cos(\omega_0 t) + 2\cos(2\omega_0 t)$$

³<http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html>

⁴"Periodic Signals" <<http://cnx.org/content/m10744/latest/>>

3 Fourier Coefficients

In general $f(t)$, the Fourier coefficients can be calculated from (2) by solving for c_n , which requires a little algebraic manipulation (for the complete derivation see the Fourier coefficients derivation⁵). The end results will yield the following general equation for the Fourier coefficients:

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-i\omega_0 n t} dt \quad (3)$$

The sequence of complex numbers $\{\forall n, n \in \mathbb{Z} : (c_n)\}$ is just an alternate representation of the function $f(t)$. Knowing the Fourier coefficients c_n is the same as knowing $f(t)$ explicitly and vice versa. Given a periodic function, we can **transform** it into its Fourier series representation using (3). Likewise, we can **inverse transform** a given sequence of complex numbers, c_n , using (2) to reconstruct the function $f(t)$.

Along with being a natural representation for signals being manipulated by LTI systems, the Fourier series provides a description of periodic signals that is convenient in many ways. By looking at the Fourier series of a signal $f(t)$, we can infer mathematical properties of $f(t)$ such as smoothness, existence of certain symmetries, as well as the physically meaningful frequency content.

3.1 Example: Using Fourier Coefficient Equation

Here we will look at a rather simple example that almost requires the use of (3) to solve for the Fourier coefficients. Once you understand the formula, the solution becomes a straightforward calculus problem. Find the Fourier coefficients for the following equation:

Exercise 4

(Solution on p. 4.)

$$f(t) = \begin{cases} 1 & \text{if } |t| \leq T \\ 0 & \text{otherwise} \end{cases}$$

4 Summary: Fourier Series Equations

Our first equation (2) is the **synthesis** equation, which builds our function, $f(t)$, by combining sinusoids.

Synthesis

$$f(t) = \sum_{n=-\infty}^{\infty} (c_n e^{i\omega_0 n t}) \quad (4)$$

And our second equation (3), termed the **analysis** equation, reveals how much of each sinusoid is in $f(t)$.

Analysis

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-i\omega_0 n t} dt \quad (5)$$

where we have stated that $\omega_0 = \frac{2\pi}{T}$.

NOTE: Understand that our interval of integration does not have to be $[0, T]$ in our Analysis Equation. We could use any interval $[a, a + T]$ of length T .

Example 1

This demonstration lets you synthesize a signal by combining sinusoids, similar to the synthesis equation for the Fourier series. See here⁶ for instructions on how to use the demo.

[MEDIA OBJECT]⁷

⁵"Derivation of Fourier Coefficients Equation" <<http://cnx.org/content/m10733/latest/>>

⁶"How to use the LabVIEW demos" <<http://cnx.org/content/m11550/latest/>>

⁷This media object is a LabVIEW VI. Please view or download it at <<http://cnx.org/content/m10496/2.22/FourierCompManip.llb>>

Solutions to Exercises in this Module

Solution to Exercise 1 (p. 2)

The tricky part of the problem is finding a way to represent the above function in terms of its basis, $e^{i\omega_0 t}$. To do this, we will use our knowledge of Euler's Relation⁸ to represent our cosine function in terms of the exponential.

$$f(t) = \frac{1}{2} \left(e^{i\omega_0 t} + e^{-i\omega_0 t} \right)$$

Now from this form of our function and from (2), by inspection we can see that our fourier coefficients will be:

$$c_n = \begin{cases} \frac{1}{2} & \text{if } |n| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution to Exercise 2 (p. 2)

As done in the previous example, we will again use Euler's Relation⁹ to represent our sine function in terms of exponential functions.

$$f(t) = \frac{1}{2i} \left(e^{i\omega_0 t} - e^{-i\omega_0 t} \right)$$

And so our fourier coefficients are

$$c_n = \begin{cases} \frac{-i}{2} & \text{if } n = -1 \\ \frac{i}{2} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution to Exercise 3 (p. 2)

Once again we will use the same technique as was used in the previous two problems. The break down of our function yields

$$f(t) = 3 + 4 \left(\frac{1}{2} \right) \left(e^{i\omega_0 t} + e^{-i\omega_0 t} \right) + 2 \left(\frac{1}{2} \right) \left(e^{i2\omega_0 t} + e^{-i2\omega_0 t} \right)$$

And from this we can find our fourier coefficients to be:

$$c_n = \begin{cases} 3 & \text{if } n = 0 \\ 2 & \text{if } |n| = 1 \\ 1 & \text{if } |n| = 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution to Exercise 4 (p. 3)

We will begin by plugging our above function, $f(t)$, into (3). Our interval of integration will now change to match the interval specified by the function.

$$c_n = \frac{1}{T} \int_{-T_1}^{T_1} (1) e^{-i\omega_0 n t} dt$$

Notice that we must consider two cases: $n = 0$ and $n \neq 0$. For $n = 0$ we can tell by inspection that we will get

$$\forall n, n = 0 : \left(c_n = \frac{2T_1}{T} \right)$$

⁸"The Complex Exponential": Section Euler's Relation <http://cnx.org/content/m10060/latest/#eul_rel>

⁹"The Complex Exponential": Section Euler's Relation <http://cnx.org/content/m10060/latest/#eul_rel>

For $n \neq 0$, we will need to take a few more steps to solve. We can begin by looking at the basic integral of the exponential we have. Remembering our calculus, we are ready to integrate:

$$c_n = \frac{1}{T} \left(\frac{1}{i\omega_0 n} \right) e^{-(i\omega_0 n t)} \Big|_{t=-T_1}^{T_1}$$

Let us now evaluate the exponential functions for the given limits and expand our equation to:

$$c_n = \frac{1}{T} \left(\frac{1}{-i\omega_0 n} \right) \left(e^{-(i\omega_0 n T_1)} - e^{i\omega_0 n T_1} \right)$$

Now if we multiple the right side of our equation by $\frac{2i}{2i}$ and distribute our negative sign into the parenthesis, we can utilize Euler's Relation¹⁰ to greatly simplify our expression into:

$$c_n = \frac{1}{T} \left(\frac{2i}{i\omega_0 n} \right) \sin(\omega_0 n T_1)$$

Now, recall earlier that we defined $\omega_0 = \frac{2\pi}{T}$. We can solve this equation for T and substitute in.

$$c_n = \frac{2i\omega_0}{i\omega_0 n 2\pi} \sin(\omega_0 n T_1)$$

And finally, if we make a few simple cancellations we will arrive at our final answer for the Fourier coefficients of $f(t)$:

$$\forall n, n \neq 0 : \left(c_n = \frac{\sin(\omega_0 n T_1)}{n\pi} \right)$$

¹⁰"The Complex Exponential": Section Euler's Relation <http://cnx.org/content/m10060/latest/#eul_rel>