Fourier Series: Eigenfunction Approach*

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Abstract

This module will introduce the Fourier Series and its Fourier coefficients using the concepts of eigenfunctions and basis. We will show several examples of how to decompose a signal and find the Fourier coefficients.

1 Introduction

Since complex exponentials[1] are eigenfunctions of linear time-invariant (LTI) systems[2], calculating the output of an LTI system $\mathcal{H}$ given e^{st} as an input amounts to simple multiplcation, where $H(s) \in \mathbb{C}$ is a constant (that depends on s). In the figure (Figure 1) below we have a simple exponential input that yields the following output:

$$y(t) = H(s)\, e^{st} \tag{1}$$

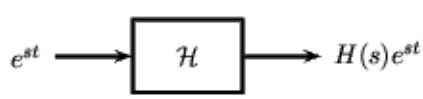

Figure 1: Simple LTI system.

Using this and the fact that $\mathcal{H}$ is linear, calculating $y(t)$ for combinations of complex exponentials is also straightforward. This linearity property is depicted in the two equations below - showing the input to the linear system H on the left side and the output, $y(t)$, on the right:

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[1] "Continuous Time Complex Exponential" <http://cnx.org/content/m10060/latest/>

[2] "Eigenfunctions of LTI Systems" <http://cnx.org/content/m10500/latest/>

1. $$c_1 e^{s_1 t} + c_2 e^{s_2 t} \rightarrow c_1 H(s_1) e^{s_1 t} + c_2 H(s_2) e^{s_2 t}$$

2. $$\sum_n c_n e^{s_n t} \rightarrow \sum_n c_n H(s_n) e^{s_n t}$$

The action of H on an input such as those in the two equations above is easy to explain: $\mathcal{H}$ **independently scales** each exponential component $e^{s_n t}$ by a different complex number $H(s_n) \in \mathbb{C}$. As such, if we can write a function $f(t)$ as a combination of complex exponentials it allows us to:

- easily calculate the output of $\mathcal{H}$ given $f(t)$ as an input (provided we know the eigenvalues $H(s)$)
- interpret how $\mathcal{H}$ manipulates $f(t)$

2 Fourier Series

Joseph Fourier[3] demonstrated that an arbitrary T-periodic function[4] $f(t)$ can be written as a linear combination of harmonic complex sinusoids

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_0 n t} \tag{2}$$

where $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency. For almost all $f(t)$ of practical interest, there exists c_n to make (2) true. If $f(t)$ is finite energy ($f(t) \in L^2[0,T]$), then the equality in (2) holds in the sense of energy convergence; if $f(t)$ is continuous, then (2) holds pointwise. Also, if $f(t)$ meets some mild conditions (the Dirichlet conditions), then (2) holds pointwise everywhere except at points of discontinuity.

The c_n - called the **Fourier coefficients** - tell us "how much" of the sinusoid $e^{i\omega_0 n t}$ is in $f(t)$. (2) essentially breaks down $f(t)$ into pieces, each of which is easily processed by an LTI system (since it is an eigenfunction of **every** LTI system). Mathematically, (2) tells us that the set of harmonic complex exponentials $\left\{\forall n, n \in \mathbb{Z} : \left(e^{i\omega_0 n t}\right)\right\}$ form a basis for the space of T-periodic continuous time functions. Below are a few examples that are intended to help you think about a given signal or function, $f(t)$, in terms of its exponential basis functions.

3 Fourier Coefficients

In general $f(t)$, the Fourier coefficients can be calculated from (2) by solving for c_n, which requires a little algebraic manipulation (for the complete derivation see the Fourier coefficients derivation[5]). The end results will yield the following general equation for the fourier coefficients:

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-(i\omega_0 n t)} dt \tag{3}$$

The sequence of complex numbers $\{\forall n, n \in \mathbb{Z} : (c_n)\}$ is just an alternate representation of the function $f(t)$. Knowing the Fourier coefficients c_n is the same as knowing $f(t)$ explicitly and vice versa. Given a periodic function, we can **transform** it into it Fourier series representation using (3). Likewise, we can **inverse transform** a given sequence of complex numbers, c_n, using (2) to reconstruct the function $f(t)$.

Along with being a natural representation for signals being manipulated by LTI systems, the Fourier series provides a description of periodic signals that is convenient in many ways. By looking at the Fourier series of a signal $f(t)$, we can infer mathematical properties of $f(t)$ such as smoothness, existence of certain symmetries, as well as the physically meaningful frequency content.

[3] http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html
[4] "Continuous Time Periodic Signals" <http://cnx.org/content/m10744/latest/>
[5] "Derivation of Fourier Coefficients Equation" <http://cnx.org/content/m10733/latest/>

4 Summary: Fourier Series Equations

Our first equation (2) is the **synthesis** equation, which builds our function, $f(t)$, by combining sinusoids.

Synthesis

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_0 nt} \tag{4}$$

And our second equation (3), termed the **analysis** equation, reveals how much of each sinusoid is in $f(t)$.

Analysis

$$c_n = \frac{1}{T} \int_0^T f(t)\, e^{-(i\omega_0 nt)} dt \tag{5}$$

where we have stated that $\omega_0 = \frac{2\pi}{T}$.

NOTE: Understand that our interval of integration does not have to be $[0, T]$ in our Analysis Equation. We could use any interval $[a, a+T]$ of length T.

Example 1

This demonstration lets you synthesize a signal by combining sinusoids, similar to the synthesis equation for the Fourier series. See here[6] for instructions on how to use the demo.

[MEDIA OBJECT][7]

[6] "How to use the LabVIEW demos" <http://cnx.org/content/m11550/latest/>

[7] This media object is a LabVIEW VI. Please view or download it at <http://cnx.org/content/m10496/2.23/FourierCompManip.llb>