EIGENFUNCTIONS OF LTI SYSTEMS*

Justin Romberg

Abstract

An introduction to eigenvalues and eigenfunctions for Linear Time Invariant systems.

1 Introduction

Hopefully you are familiar with the notion of the eigenvectors of a "matrix system," if not they do a quick review of eigen-stuff[1]. We can develop the same ideas for LTI systems acting on signals. A linear time invariant (LTI) system[2]$\mathcal{H}$ operating on a continuous input $f(t)$ to produce continuous time output $y(t)$

$$\mathcal{H}[f(t)] = y(t) \quad (1)$$

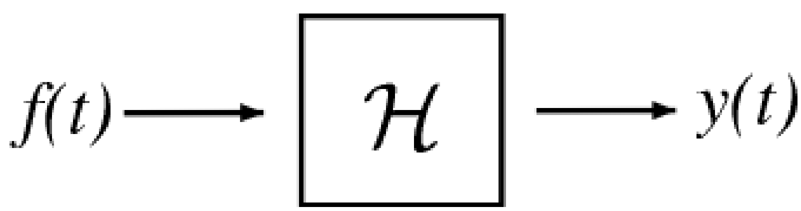

Figure 1: $\mathcal{H}[f(t)] = y(t)$. f and t are continuous time (CT) signals and $\mathcal{H}$ is an LTI operator.

is mathematically analogous to an NxN matrix A operating on a vector $\mathbf{x} \in \mathbb{C}^N$ to produce another vector $\mathbf{b} \in \mathbb{C}^N$ (see Matrices and LTI Systems for an overview).

$$\mathbf{Ax} = \mathbf{b} \quad (2)$$

*Version 2.8: Sep 30, 2009 3:34 pm GMT-5

[1]"Eigen-stuff in a Nutshell" <http://cnx.org/content/m10742/latest/>
[2]"Introduction to Systems" <http://cnx.org/content/m0005/latest/>

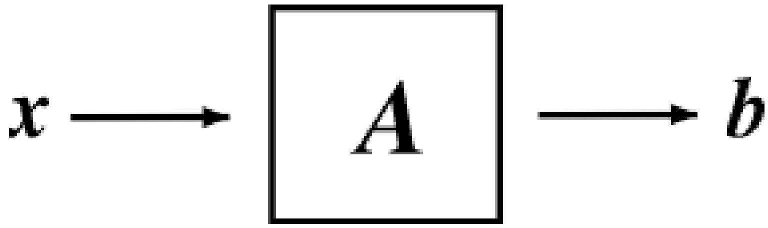

Figure 2: $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{x}$ and $\mathbf{b}$ are in $\mathbb{C}^N$ and $\mathbf{A}$ is an N x N matrix.

Just as an eigenvector[3] of $\mathbf{A}$ is a $\mathbf{v} \in \mathbb{C}^N$ such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, $\lambda \in \mathbb{C}$,

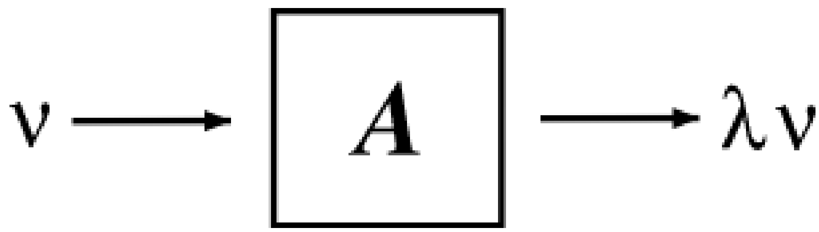

Figure 3: $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ where $\mathbf{v} \in \mathbb{C}^N$ is an eigenvector of $\mathbf{A}$.

we can define an **eigenfunction** (or **eigensignal**) of an LTI system $\mathcal{H}$ to be a signal $f(t)$ such that

$$\forall \lambda, \lambda \in \mathbb{C} : (\mathcal{H}[f(t)] = \lambda f(t)) \quad (3)$$

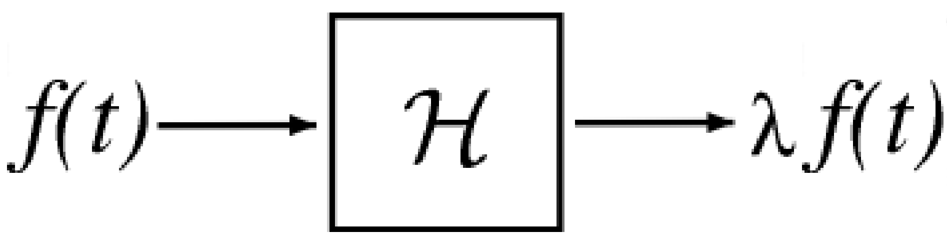

Figure 4: $\mathcal{H}[f(t)] = \lambda f(t)$ where f is an eigenfunction of $\mathcal{H}$.

Eigenfunctions are the **simplest** possible signals for $\mathcal{H}$ to operate on: to calculate the output, we simply multiply the input by a complex number λ.

[3]"Eigenvectors and Eigenvalues" <http://cnx.org/content/m10736/latest/>

2 Eigenfunctions of any LTI System

The class of LTI systems has a set of eigenfunctions in common: the complex exponentials[4] e^{st}, $s \in \mathbb{C}$ are eigenfunctions for **all** LTI systems.

$$\mathcal{H}\left[e^{st}\right] = \lambda_s e^{st} \quad (4)$$

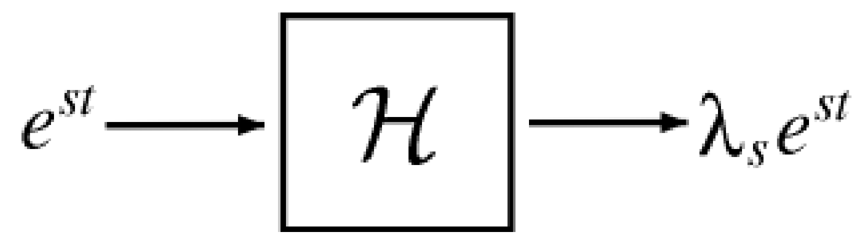

Figure 5: $\mathcal{H}\left[e^{st}\right] = \lambda_s e^{st}$ where $\mathcal{H}$ is an LTI system.

NOTE: While $\{\forall s, s \in \mathbb{C} : (e^{st})\}$ are always eigenfunctions of an LTI system, they are not necessarily the **only** eigenfunctions.

We can prove (4) by expressing the output as a convolution[5] of the input e^{st} and the impulse response[6] $h(t)$ of $\mathcal{H}$:

$$\begin{aligned} \mathcal{H}[e^{st}] &= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)} d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\, e^{st} e^{-(s\tau)} d\tau \\ &= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-(s\tau)} d\tau \end{aligned} \quad (5)$$

Since the expression on the right hand side does not depend on t, it is a constant, λ_s. Therefore

$$\mathcal{H}\left[e^{st}\right] = \lambda_s e^{st} \quad (6)$$

The eigenvalue λ_s is a complex number that depends on the exponent s and, of course, the system $\mathcal{H}$. To make these dependencies explicit, we will use the notation $H(s) \equiv \lambda_s$.

[4]"The Complex Exponential" <http://cnx.org/content/m10060/latest/>
[5]"Continuous-Time Convolution" <http://cnx.org/content/m10085/latest/>
[6]"The Impulse Function" <http://cnx.org/content/m10059/latest/>

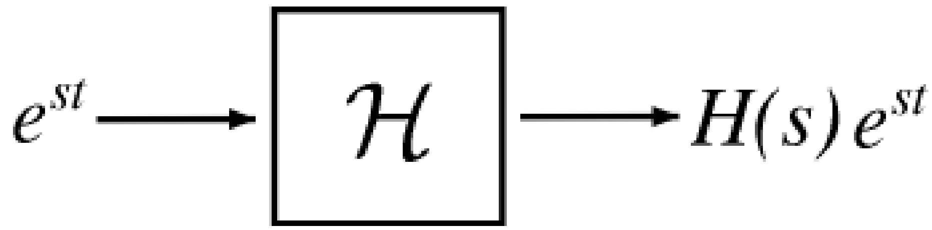

Figure 6: e^{st} is the eigenfunction and $H(s)$ are the eigenvalues.

Since the action of an LTI operator on its eigenfunctions e^{st} is easy to calculate and interpret, it is convenient to represent an arbitrary signal $f(t)$ as a linear combination of complex exponentials. The Fourier series[7] gives us this representation for periodic continuous time signals, while the (slightly more complicated) Fourier transform[8] lets us expand arbitrary continuous time signals.

[7]"Fourier Series: Eigenfunction Approach" <http://cnx.org/content/m10496/latest/>
[8]"Derivation of the Fourier Transform" <http://cnx.org/content/m0046/latest/>