The Z Transform: Definition*

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Abstract

A brief definition of the z-transform, explaining its relationship with the Fourier transform and its region of convergence, ROC.

1 Basic Definition of the Z-Transform

The **z-transform** of a sequence is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{1}$$

Sometimes this equation is referred to as the **bilateral z-transform**. At times the z-transform is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \tag{2}$$

which is known as the **unilateral z-transform**.

There is a close relationship between the z-transform and the **Fourier transform** of a discrete time signal, which is defined as

$$X\left(e^{i\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-(i\omega n)} \tag{3}$$

Notice that that when the z^{-n} is replaced with $e^{-(i\omega n)}$ the z-transform reduces to the Fourier Transform. When the Fourier Transform exists, $z = e^{i\omega}$, which is to have the magnitude of z equal to unity.

2 The Complex Plane

In order to get further insight into the relationship between the Fourier Transform and the Z-Transform it is useful to look at the complex plane or **z-plane**. Take a look at the complex plane:

*Version 2.10: Jul 29, 2010 2:47 pm -0500

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Z-Plane

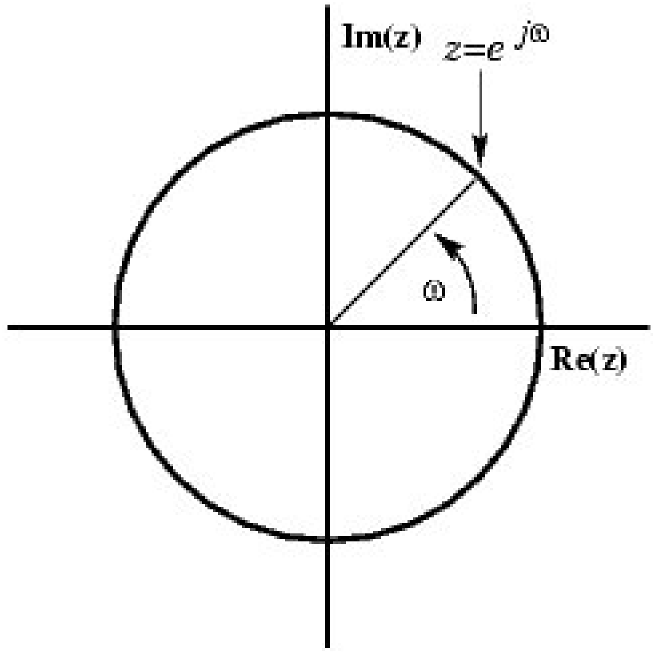

Figure 1

The Z-plane is a complex plane with an imaginary and real axis referring to the complex-valued variable z. The position on the complex plane is given by $re^{i\omega}$, and the angle from the positive, real axis around the plane is denoted by ω. $X(z)$ is defined everywhere on this plane. $X\left(e^{i\omega}\right)$ on the other hand is defined only where $|z| = 1$, which is referred to as the unit circle. So for example, $\omega = 1$ at $z = 1$ and $\omega = \pi$ at $z = -1$. This is useful because, by representing the Fourier transform as the z-transform on the unit circle, the periodicity of Fourier transform is easily seen.

3 Region of Convergence

The region of convergence, known as the **ROC**, is important to understand because it defines the region where the z-transform exists. The ROC for a given $x[n]$, is defined as the range of z for which the z-transform converges. Since the z-transform is a **power series**, it converges when $x[n]\,z^{-n}$ is absolutely summable. Stated differently,

$$\sum_{n=-\infty}^{\infty} \left|x[n]\,z^{-n}\right| < \infty \tag{4}$$

must be satisfied for convergence. This is best illustrated by looking at the different ROC's of the z-transforms of $\alpha^n u[n]$ and $\alpha^n u[n-1]$.

Example 1
For

$$x[n] = \alpha^n u[n] \tag{5}$$

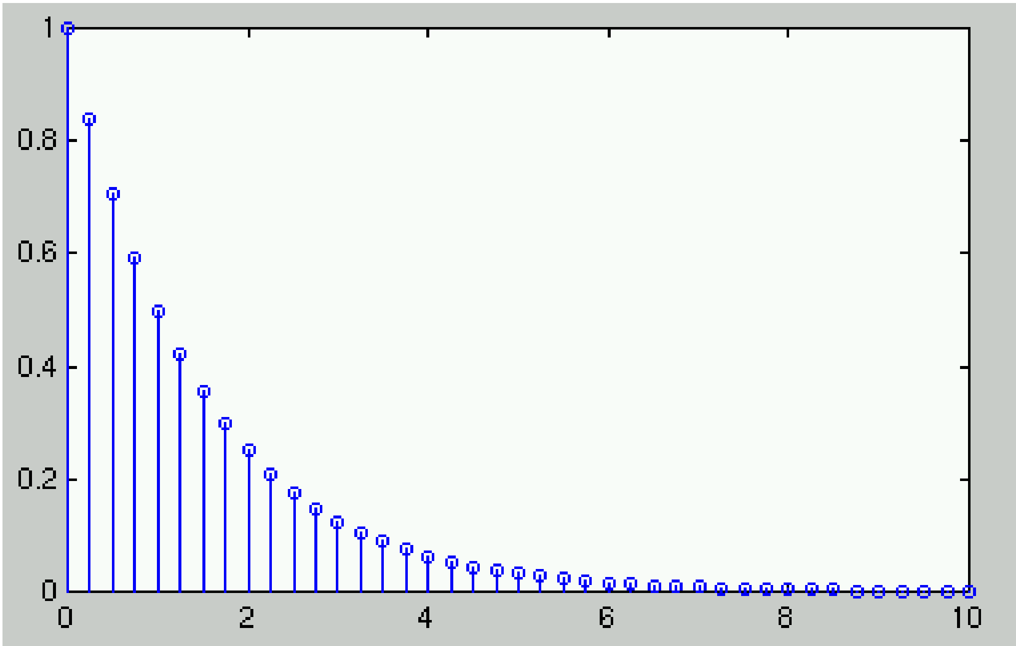

Figure 2: $x[n] = \alpha^n u[n]$ where $\alpha = 0.5$.

$$\begin{array}{rcl} X(z) & = & \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ & = & \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} \\ & = & \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ & = & \sum_{n=0}^{\infty} \left(\alpha z^{-1}\right)^n \end{array} \quad (6)$$

This sequence is an example of a right-sided exponential sequence because it is nonzero for $n \geq 0$. It only converges when $|\alpha z^{-1}| < 1$. When it converges,

$$\begin{array}{rcl} X(z) & = & \frac{1}{1-\alpha z^{-1}} \\ & = & \frac{z}{z-\alpha} \end{array} \quad (7)$$

If $|\alpha z^{-1}| \geq 1$, then the series, $\sum_{n=0}^{\infty} \left(\alpha z^{-1}\right)^n$ does not converge. Thus the ROC is the range of values where

$$|\alpha z^{-1}| < 1 \quad (8)$$

or, equivalently,

$$|z| > |\alpha| \quad (9)$$

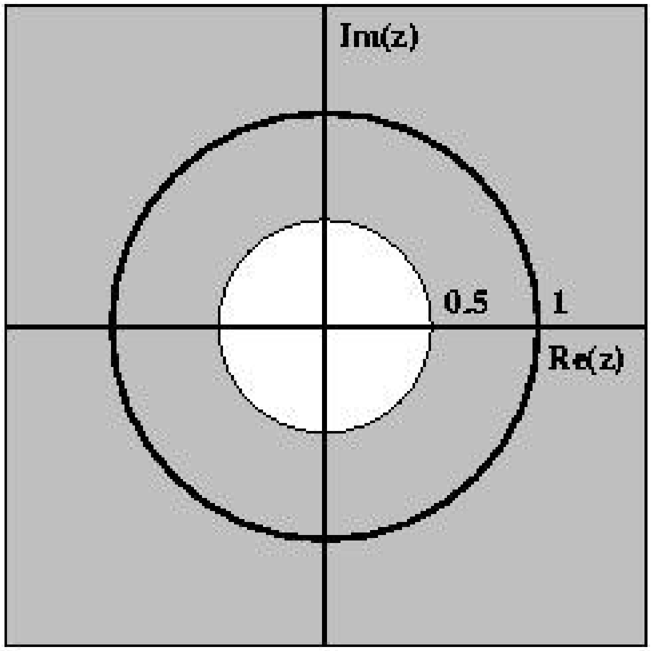

Figure 3: ROC for $x[n] = \alpha^n u[n]$ where $\alpha = 0.5$

Example 2
For

$$x[n] = (-\alpha^n)\, u[(-n) - 1] \tag{10}$$

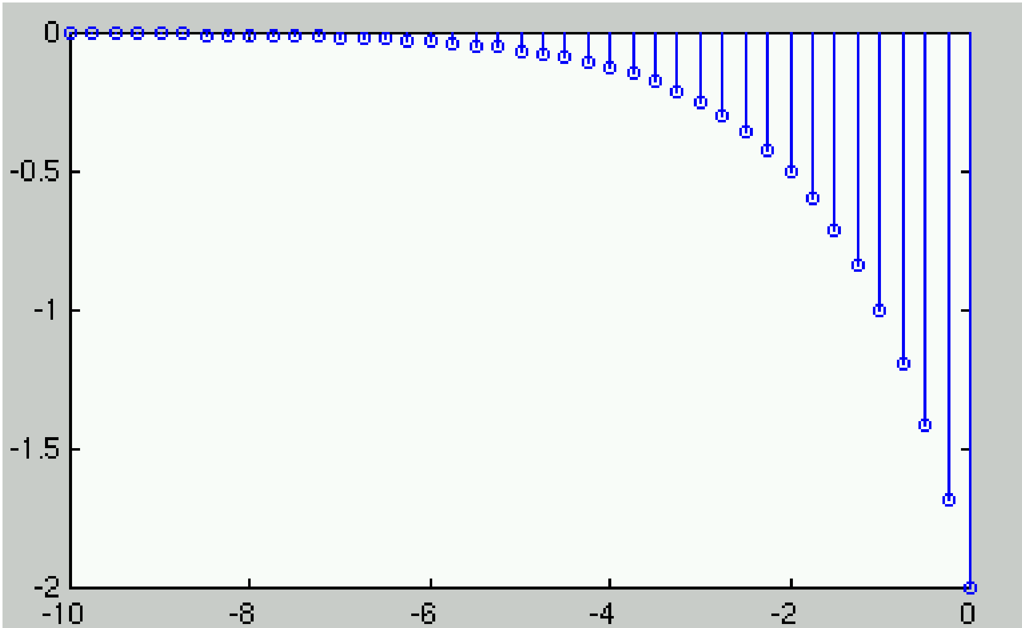

Figure 4: $x[n] = (-\alpha^n)\, u[(-n)-1]$ where $\alpha = 0.5$.

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \\ &= \sum_{n=-\infty}^{\infty} (-\alpha^n)\, u[-\mathrm{n}-1]\, z^{-n} \\ &= -\sum_{n=-\infty}^{-1} \alpha^n z^{-n} \\ &= -\sum_{n=-\infty}^{-1} \left(\alpha^{-1} z\right)^{-n} \\ &= -\sum_{n=1}^{\infty} \left(\alpha^{-1} z\right)^{n} \\ &= 1 - \sum_{n=0}^{\infty} \left(\alpha^{-1} z\right)^{n} \end{aligned} \quad (11)$$

The ROC in this case is the range of values where

$$|\alpha^{-1} z| < 1 \quad (12)$$

or, equivalently,

$$|z| < |\alpha| \quad (13)$$

If the ROC is satisfied, then

$$\begin{aligned} X(z) &= 1 - \frac{1}{1-\alpha^{-1} z} \\ &= \frac{z}{z-\alpha} \end{aligned} \quad (14)$$

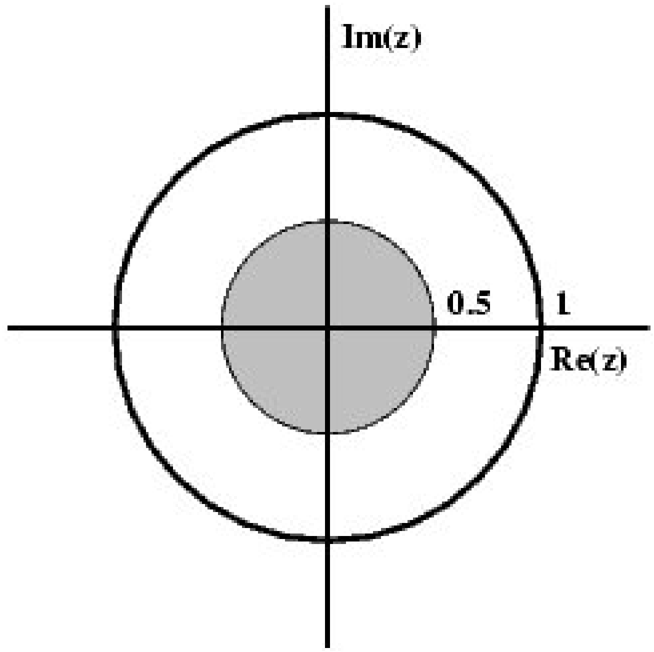

Figure 5: ROC for $x[n] = (-\alpha^n)\, u[(-n) - 1]$