

AUTOCORRELATION OF RANDOM PROCESSES*

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Abstract

The module will explain Autocorrelation and its function and properties. Also, examples will be provided to help you step through some of the more complicated statistical analysis.

Before diving into a more complex statistical analysis of random signals and processes¹, let us quickly review the idea of correlation². Recall that the correlation of two signals or variables is the expected value of the product of those two variables. Since our focus will be to discover more about a random process, a collection of random signals, then imagine us dealing with two samples of a random process, where each sample is taken at a different point in time. Also recall that the key property of these random processes is that they are now functions of time; imagine them as a collection of signals. The expected value³ of the product of these two variables (or samples) will now depend on how quickly they change in regards to **time**. For example, if the two variables are taken from almost the same time period, then we should expect them to have a high correlation. We will now look at a correlation function that relates a pair of random variables from the same process to the time separations between them, where the argument to this correlation function will be the time difference. For the correlation of signals from two different random process, look at the crosscorrelation function⁴.

1 Autocorrelation Function

The first of these correlation functions we will discuss is the **autocorrelation**, where each of the random variables we will deal with come from the same random process.

Definition 1: Autocorrelation

the expected value of the product of a random variable or signal realization with a time-shifted version of itself

With a simple calculation and analysis of the autocorrelation function, we can discover a few important characteristics about our random process. These include:

1. How quickly our random signal or processes changes with respect to the time function
2. Whether our process has a periodic component and what the expected frequency might be

*Version 2.4: Apr 5, 2005 8:51 pm -0500

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¹"Introduction to Random Signals and Processes" <<http://cnx.org/content/m10649/latest/>>

²"Correlation and Covariance of a Random Signal" <<http://cnx.org/content/m10673/latest/>>

³"Random Processes: Mean and Variance" <<http://cnx.org/content/m10656/latest/>>

⁴"Crosscorrelation of Random Processes" <<http://cnx.org/content/m10686/latest/>>

As was mentioned above, the autocorrelation function is simply the expected value of a product. Assume we have a pair of random variables from the same process, $X_1 = X(t_1)$ and $X_2 = X(t_2)$, then the autocorrelation is often written as

$$\begin{aligned} R_{xx}(t_1, t_2) &= E[X_1 X_2] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2) dx_2 dx_1 \end{aligned} \quad (1)$$

The above equation is valid for stationary and nonstationary random processes. For stationary processes⁵, we can generalize this expression a little further. Given a wide-sense stationary processes, it can be proven that the expected values from our random process will be independent of the origin of our time function. Therefore, we can say that our autocorrelation function will depend on the time difference and not some absolute time. For this discussion, we will let $\tau = t_2 - t_1$, and thus we generalize our autocorrelation expression as

$$\begin{aligned} R_{xx}(t, t + \tau) &= R_{xx}(\tau) \\ &= E[X(t) X(t + \tau)] \end{aligned} \quad (2)$$

for the continuous-time case. In most DSP course we will be more interested in dealing with real signal sequences, and thus we will want to look at the discrete-time case of the autocorrelation function. The formula below will prove to be more common and useful than (1):

$$R_{xx}[n, n + m] = \sum_{n=-\infty}^{\infty} x[n] x[n + m] \quad (3)$$

And again we can generalize the notation for our autocorrelation function as

$$\begin{aligned} R_{xx}[n, n + m] &= R_{xx}[m] \\ &= E[X[n] X[n + m]] \end{aligned} \quad (4)$$

1.1 Properties of Autocorrelation

Below we will look at several properties of the autocorrelation function that hold for **stationary** random processes.

- Autocorrelation is an even function for τ

$$R_{xx}(\tau) = R_{xx}(-\tau)$$

- The mean-square value can be found by evaluating the autocorrelation where $\tau = 0$, which gives us

$$R_{xx}(0) = \overline{X^2}$$

- The autocorrelation function will have its largest value when $\tau = 0$. This value can appear again, for example in a periodic function at the values of the equivalent periodic points, but will never be exceeded.

$$R_{xx}(0) \geq |R_{xx}(\tau)|$$

- If we take the autocorrelation of a period function, then $R_{xx}(\tau)$ will also be periodic with the same frequency.

⁵"Stationary and Nonstationary Random Processes" <<http://cnx.org/content/m10684/latest/>>

1.2 Estimating the Autocorrelation with Time-Averaging

Sometimes the whole random process is not available to us. In these cases, we would still like to be able to find out some of the characteristics of the stationary random process, even if we just have part of one sample function. In order to do this we can **estimate** the autocorrelation from a given interval, 0 to T seconds, of the sample function.

$$\check{R}_{xx}(\tau) = \frac{1}{T - \tau} \int_0^{T-\tau} x(t) x(t + \tau) dt \quad (5)$$

However, a lot of times we will not have sufficient information to build a complete continuous-time function of one of our random signals for the above analysis. If this is the case, we can treat the information we do know about the function as a discrete signal and use the discrete-time formula for estimating the autocorrelation.

$$\check{R}_{xx}[m] = \frac{1}{N - m} \sum_{n=0}^{N-m-1} x[n] x[n + m] \quad (6)$$

2 Examples

Below we will look at a variety of examples that use the autocorrelation function. We will begin with a simple example dealing with Gaussian White Noise (GWN) and a few basic statistical properties that will prove very useful in these and future calculations.

Example 1

We will let $x[n]$ represent our GWN. For this problem, it is important to remember the following fact about the mean of a GWN function:

$$E[x[n]] = 0$$

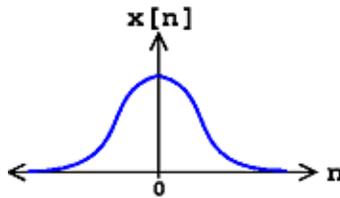


Figure 1: Gaussian density function. By examination, can easily see that the above statement is true - the mean equals zero.

Along with being **zero-mean**, recall that GWN is always **independent**. With these two facts, we are now ready to do the short calculations required to find the autocorrelation.

$$R_{xx}[n, n + m] = E[x[n] x[n + m]]$$

Since the function, $x[n]$, is independent, then we can take the product of the individual expected values of both functions.

$$R_{xx}[n, n + m] = E[x[n]] E[x[n + m]]$$

Now, looking at the above equation we see that we can break it up further into two conditions: one when m and n are equal and one when they are not equal. When they are equal we can combine

the expected values. We are left with the following piecewise function to solve:

$$R_{xx}[n, n+m] = \begin{cases} E[x[n]] E[x[n+m]] & \text{if } m \neq 0 \\ E[x^2[n]] & \text{if } m = 0 \end{cases}$$

We can now solve the two parts of the above equation. The first equation is easy to solve as we have already stated that the expected value of $x[n]$ will be zero. For the second part, you should recall from statistics that the expected value of the square of a function is equal to the variance. Thus we get the following results for the autocorrelation:

$$R_{xx}[n, n+m] = \begin{cases} 0 & \text{if } m \neq 0 \\ \sigma^2 & \text{if } m = 0 \end{cases}$$

Or in a more concise way, we can represent the results as

$$R_{xx}[n, n+m] = \sigma^2 \delta[m]$$