Linear-Phase FIR Filters*

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1 THE AMPLITUDE RESPONSE

If the real and imaginary parts of $H^f(\omega)$ are given by

$$H^f(\omega) = \Re(\omega) + i\Im(\omega) \tag{1}$$

the magnitude and phase are defined as

$$|H^f(\omega)| = \sqrt{(\Re(\omega))^2 + (\Im(\omega))^2}$$

$$p(\omega) = \arctan\left(\frac{\Im(\omega)}{\Re(\omega)}\right)$$

so that

$$H^f(\omega) = |H^f(\omega)| e^{ip(\omega)} \tag{2}$$

With this definition, $|H^f(\omega)|$ is never negative and $p(\omega)$ is usually discontinuous, but it can be very helpful to write $H^f(\omega)$ as

$$H^f(\omega) = A(\omega) e^{i\theta(\omega)} \tag{3}$$

where $A(\omega)$ can be positive and negative, and $\theta(\omega)$ continuous. $A(\omega)$ is called the **amplitude response**. Figure 1 illustrates the difference between $|H^f(\omega)|$ and $A(\omega)$.

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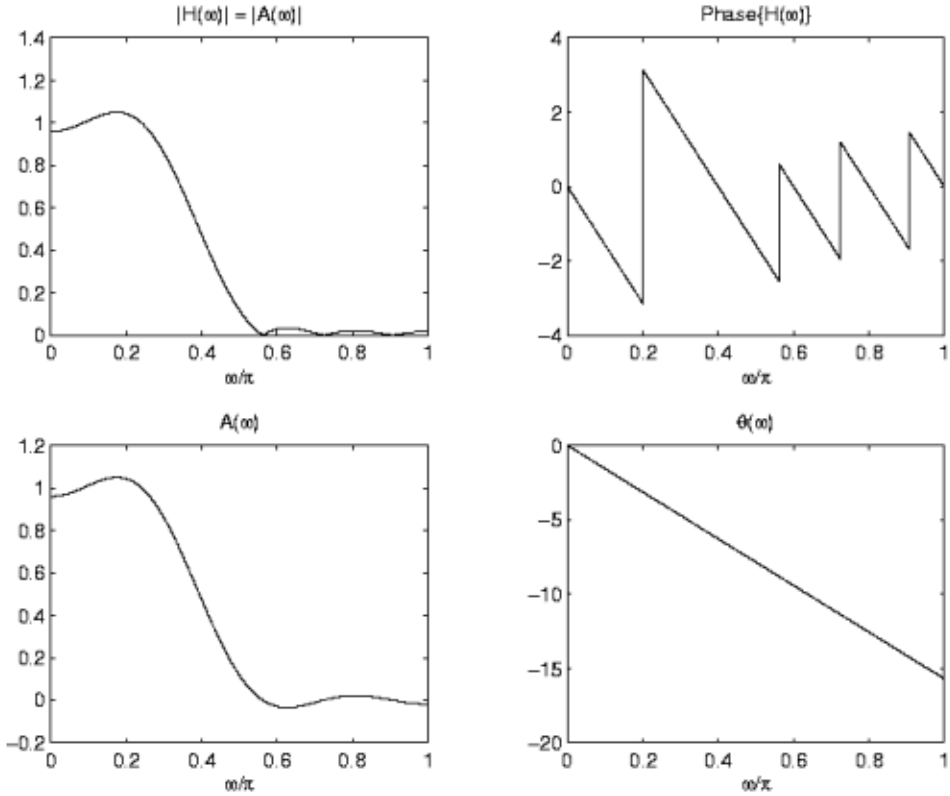

Figure 1

A linear-phase phase filter is one for which the continuous phase $\theta(\omega)$ is linear.

$$H^f(\omega) = A(\omega) e^{i\theta(\omega)}$$

with

$$\theta(\omega) = M\omega + B$$

We assume in the following that the impulse response $h(n)$ is real-valued.

2 WHY LINEAR-PHASE?

If a discrete-time cosine signal

$$x_1(n) = \cos(\omega_1 n + \phi_1)$$

is processed through a discrete-time filter with frequency response

$$H^f(\omega) = A(\omega) e^{i\theta(\omega)}$$

then the output signal is given by

$$y_1(n) = A(\omega_1) \cos(\omega_1 n + \phi_1 + \theta(\omega_1))$$

or

$$y_1(n) = A(\omega_1)\cos\left(\omega_1\left(n + \frac{\theta(\omega_1)}{\omega_1}\right) + \phi_1\right)$$

The LTI system has the effect of scaling the cosine signal and delaying it by $\frac{\theta(\omega_1)}{\omega_1}$.

Exercise 1 ***(Solution on p. 6.)***

When does the system delay cosine signals with different frequencies by the same amount?

The function $\frac{\theta(\omega)}{\omega}$ is called the **phase delay**. A linear phase filter therefore has constant phase delay.

3 WHY LINEAR-PHASE: EXAMPLE

Consider a discrete-time filter described by the difference equation

$$y(n) = 0.1821x(n) + 0.7865x(n-1) - 0.6804x(n-2) + x(n-3) + 0.6804y(n-1) - 0.7865y(n-2) + 0.1821y(n-3) \quad (4)$$

When $\omega_1 = 0.31\pi$, then the delay is $\frac{-\theta(\omega_1)}{\omega_1} = 2.45$. The delay is illustrated in Figure 2:

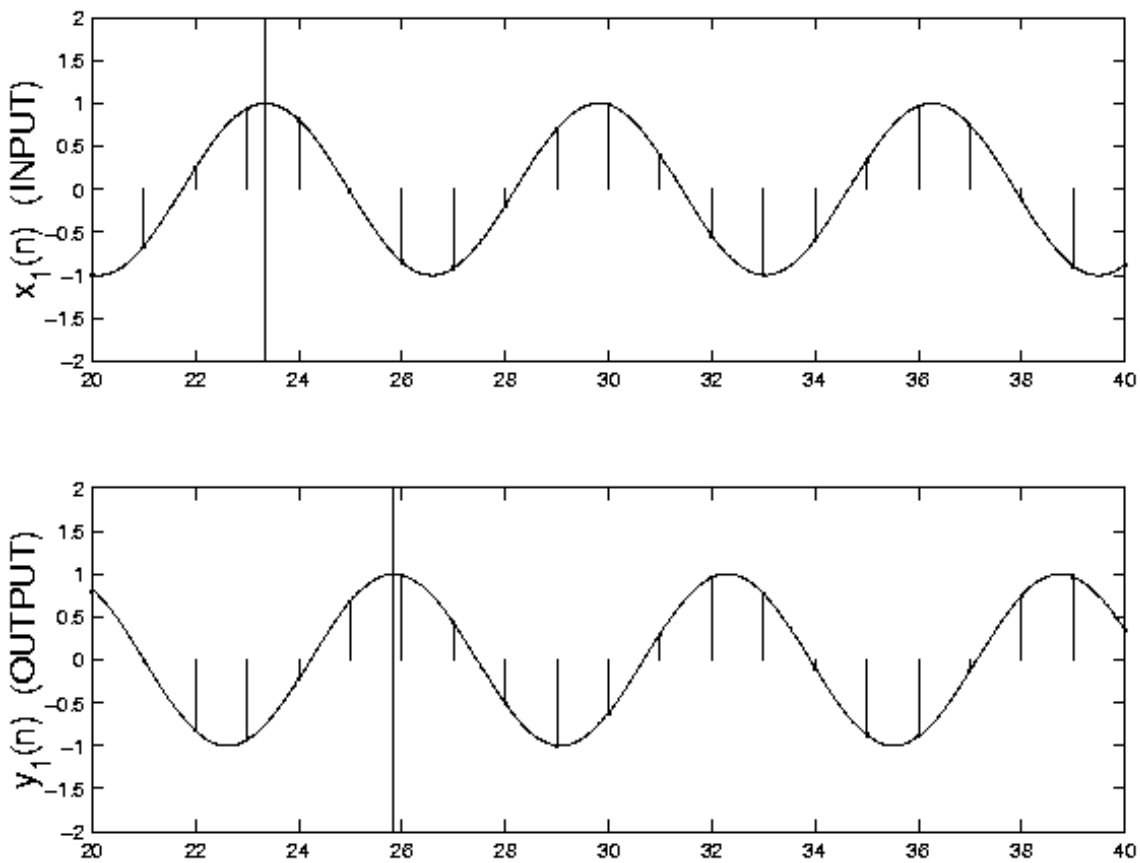

Figure 2

Notice that the delay is fractional — the discrete-time samples are not exactly reproduced in the output. The fractional delay can be interpreted in this case as a delay of the underlying continuous-time cosine signal.

4 WHY LINEAR-PHASE: EXAMPLE (2)

Consider the same system given on the previous slide, but let us change the frequency of the cosine signal.

When $\omega_2 = 0.47\pi$, then the delay is $\frac{-\theta(\omega_2)}{\omega_2} = 0.14$.

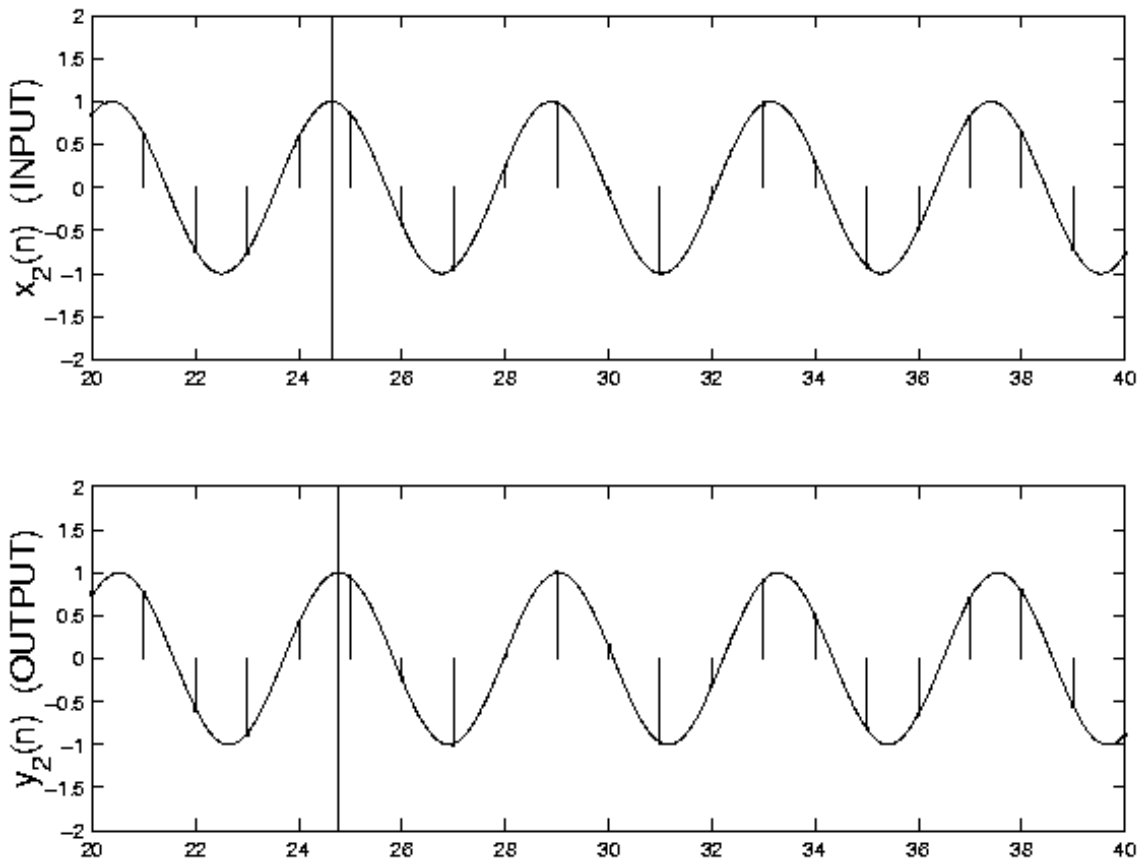

Figure 3

NOTE: For this example, the delay depends on the frequency, because this system does not have linear phase.

5 WHY LINEAR-PHASE: MORE

From the previous slides, we see that a filter will delay different frequency components of a signal by the same amount if the filter has linear phase (constant phase delay).

In addition, when a narrow band signal (as in AM modulation) goes through a filter, the envelop will be delayed by the **group delay** or **envelop delay** of the filter. The amount by which the envelop is delayed is independent of the carrier frequency only if the filter has linear phase.

Also, in applications like image processing, filters with non-linear phase can introduce artifacts that are visually annoying.

Solutions to Exercises in this Module

Solution to Exercise (p. 3)

- $\frac{\theta(\omega)}{\omega} = \text{constant}$
- $\theta(\omega) = K\omega$
- The phase is linear.