LINEAR-PHASE FIR FILTERS: AMPLITUDE FORMULAS*

Ivan Selesnick

1 SUMMARY: AMPLITUDE FORMULAS

Type	$\theta(\omega)$	$A(\omega)$
I	$-(M\omega)$	$h(M)+2\sum_{n=0}^{M-1}h(n)\cos((M-n)\omega)$
II	$-(M\omega)$	$2\sum_{n=0}^{\frac{N}{2}-1}h(n)\cos((M-n)\omega)$
III	$-(M\omega)+\frac{\pi}{2}$	$2\sum_{n=0}^{M-1}h(n)\sin((M-n)\omega)$
IV	$-(M\omega)+\frac{\pi}{2}$	$2\sum_{n=0}^{\frac{N}{2}-1}h(n)\sin((M-n)\omega)$

Table 1

where $M=\frac{N-1}{2}$

2 AMPLITUDE RESPONSE CHARACTERISTICS

To analyze or design linear-phase FIR filters, we need to know the characteristics of the amplitude response $A(\omega)$.

Type	Properties	
I	$A(\omega)$ is even about $\omega=0$	$A(\omega)=A(-\omega)$
	$A(\omega)$ is even about $\omega=\pi$	$A(\pi+\omega)=A(\pi-\omega)$
	$A(\omega)$ is periodic with 2π	$A(\omega+2\pi)=A(\omega)$
	$A(\omega)$ is even about $\omega=0$	$A(\omega)=A(-\omega)$
II		

continued on next page

*Version 2.5: Aug 9, 2005 3:18 pm -0500

	$A(\omega)$ is odd about $\omega = \pi$	$A(\pi + \omega) = -A(\pi - \omega)$
	$A(\omega)$ is periodic with 4π	$A(\omega + 4\pi) = A(\omega)$
III	$A(\omega)$ is odd about $\omega = 0$	$A(\omega) = -A(-\omega)$
	$A(\omega)$ is odd about $\omega = \pi$	$A(\pi + \omega) = -A(\pi - \omega)$
	$A(\omega)$ is periodic with 2π	$A(\omega + 2\pi) = A(\omega)$
IV	$A(\omega)$ is odd about $\omega = 0$	$A(\omega) = -A(-\omega)$
	$A(\omega)$ is even about $\omega = \pi$	$A(\pi + \omega) = A(\pi - \omega)$
	$A(\omega)$ is periodic with 4π	$A(\omega + 4\pi) = A(\omega)$

Table 2

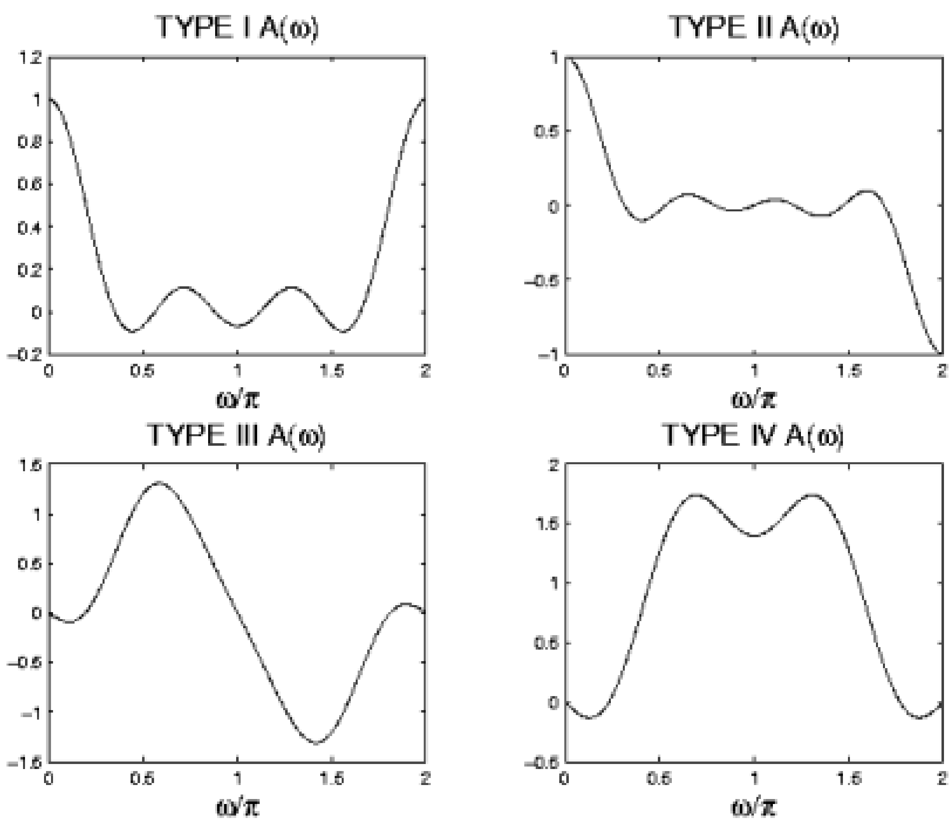

Figure 1

3 EVALUATING THE AMPLITUDE RESPONSE

The frequency response $H^f(\omega)$ of an FIR filter can be evaluated at L equally spaced frequencies between 0 and π using the DFT. Consider a causal FIR filter with an impulse response $h(n)$ of length-N, with $N \leq L$.

Samples of the frequency response of the filter can be written as

$$H\left(\frac{2\pi}{L}k\right) = \sum_{n=0}^{N-1} h(n)\, e^{(-i)\frac{2\pi}{L}nk}$$

Define the L-point signal $\{g(n) \mid 0 \leq n \leq L-1\}$ as

$$g(n) = \begin{cases} h(n) & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{if } N \leq n \leq L-1 \end{cases}$$

Then

$$H\left(\frac{2\pi}{L}k\right) = G(k) = \mathrm{DFT}_L\left(g(n)\right)$$

where $G(k)$ is the L-point DFT of $g(n)$.

3.1 Types I and II

Suppose the FIR filter $h(n)$ is either a Type I or a Type II FIR filter. Then we have from above

$$H^f(\omega) = A(\omega)\, e^{(-i)M\omega}$$

or

$$A(\omega) = H^f(\omega)\, e^{iM\omega}$$

Samples of the real-valued amplitude $A(\omega)$ can be obtained from samples of the function $H^f(\omega)$ as:

$$A\left(\frac{2\pi}{L}k\right) = H\left(\frac{2\pi}{L}k\right) e^{iM\frac{2\pi}{L}k} = G(k)\, W_L^{Mk}$$

Therefore, the samples of the real-valued amplitude function can be obtained by zero-padding $h(n)$, taking the DFT, and multiplying by the complex exponential. This can be written as:

$$A\left(\frac{2\pi}{L}k\right) = \mathrm{DFT}_L\left(\left[h(n), 0_{L-N}\right]\right) W_L^{Mk} \quad (1)$$

3.2 Types III and IV

For Type III and Type IV FIR filters, we have

$$H^f(\omega) = ie^{(-i)M\omega} A(\omega)$$

or

$$A(\omega) = (-i)\, H^f(\omega)\, e^{iM\omega}$$

Therefore, samples of the real-valued amplitude $A(\omega)$ can be obtained from samples of the function $H^f(\omega)$ as:

$$A\left(\frac{2\pi}{L}k\right) = (-i)\, H\left(\frac{2\pi}{L}k\right) e^{iM\frac{2\pi}{L}k} = (-i)\, G(k)\, W_L^{Mk}$$

Therefore, the samples of the real-valued amplitude function can be obtained by zero-padding $h(n)$, taking the DFT, and multiplying by the complex exponential.

$$A\left(\frac{2\pi}{L}k\right) = (-i)\, \mathrm{DFT}_L\left(\left[h(n), 0_{L-N}\right]\right) W_L^{Mk} \quad (2)$$

Example 1: EVALUATING THE AMP RESP (TYPE I)
In this example, the filter is a Type I FIR filter of length 7. An accurate plot of $A(\omega)$ can be obtained with zero padding.

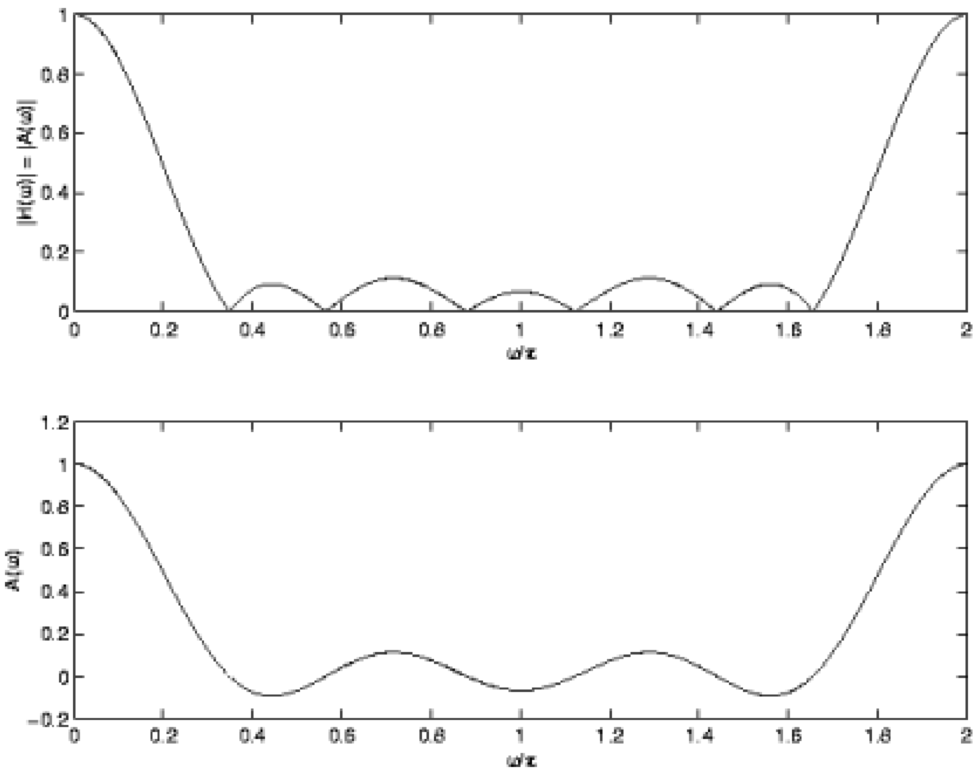

Figure 2

The following Matlab code fragment for the plot of $A(\omega)$ for a Type I FIR filter.

```
h = [3 4 5 6 5 4 3]/30;
N = 7;
M = (N-1)/2;
L = 512;
H = fft([h zeros(1,L-N)]);
k = 0:L-1;
W = exp(j*2*pi/L);
A = H .* W.^(M*k);
A = real(A);

figure(1)
w = [0:L-1]*2*pi/(L-1);
subplot(2,1,1)
plot(w/pi,abs(H))
```

```
ylabel('|H(\omega)| = |A(\omega)|')
xlabel('\omega/\pi')
subplot(2,1,2)
plot(w/pi,A)
ylabel('A(\omega)')
xlabel('\omega/\pi')
print -deps type1
```

The command `A = real(A)` removes the imaginary part which is equal to zero to within computer precision. Without this command, Matlab takes `A` to be a complex vector and the following plot command will not be right.

Observe the symmetry of $A(\omega)$ due to $h(n)$ being real-valued. Because of this symmetry, $A(\omega)$ is usually plotted for $0 \le \omega \le \pi$ only.

Example 2: EVALUATING THE AMP RESP (TYPE II)

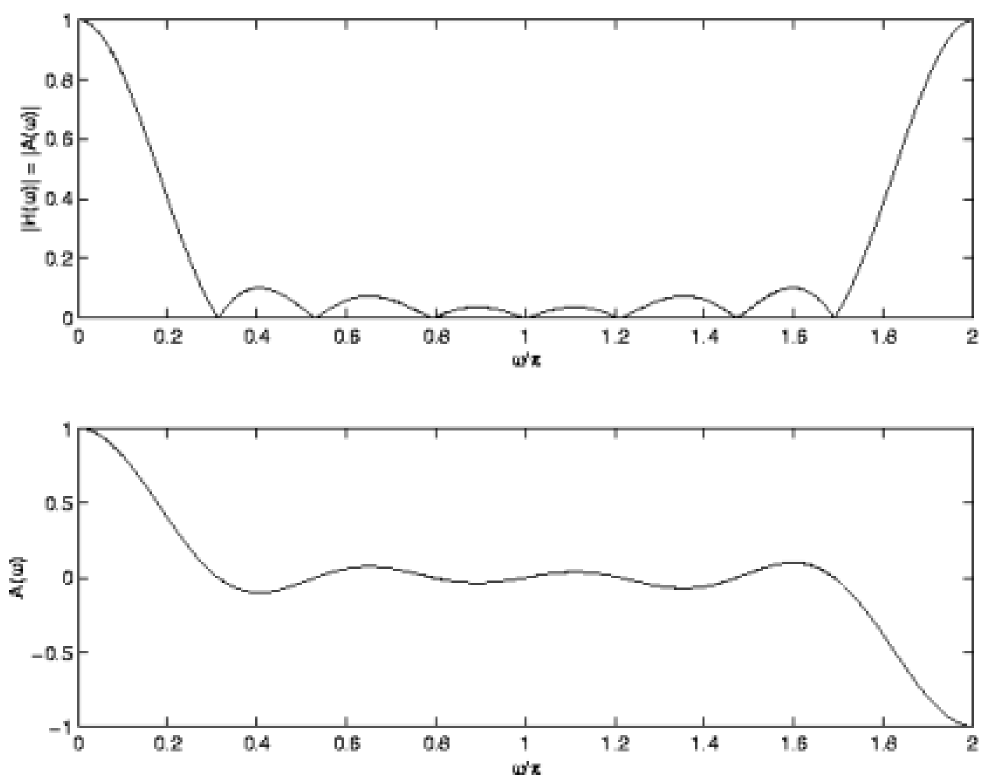

Figure 3

The following Matlab code fragment produces a plot of $A(\omega)$ for a Type II FIR filter.

```
h = [3 5 6 7 7 6 5 3]/42;
N = 8;
M = (N-1)/2;
```

```
L = 512;
H = fft([h zeros(1,L-N)]);
k = 0:L-1;
W = exp(j*2*pi/L);
A = H .* W.^(M*k);
A = real(A);

figure(1)
w = [0:L-1]*2*pi/(L-1);
subplot(2,1,1)
plot(w/pi,abs(H))
ylabel('|H(\omega)| = |A(\omega)|')
xlabel('\omega/\pi')
subplot(2,1,2)
plot(w/pi,A)
ylabel('A(\omega)')
xlabel('\omega/\pi')
print -deps type2
```

The imaginary part of the amplitude is zero. Notice that $A(\pi) = 0$. In fact this will always be the case for a Type II FIR filter.

An exercise for the student: Describe how to obtain samples of $A(\omega)$ for Type III and Type IV FIR filters. Modify the Matlab code above for these types. Do you notice that $A(\omega) = 0$ always for special values of ω?

4 Modules for Further Study

1. Zero Locations of Linear-Phase Filters[1]
2. Design of Linear-Phase FIR Filters by Interpolation[2]
3. Linear-Phase FIR Filter Design by Least Squares[3]

[1] "Zero Locations of Linear-Phase FIR Filters" <http://cnx.org/content/m10700/latest/>
[2] "Design of Linear-Phase FIR Filters by DFT-Based Interpolation" <http://cnx.org/content/m10701/latest/>
[3] "Linear-Phase Fir Filter Design By Least Squares" <http://cnx.org/content/m10577/latest/>