

FOURIER SERIES IN A NUTSHELL*

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Abstract

This module will give a brief over of the key concepts involving the Fourier series and the tools used to decompose and approximate a given signal.

1 Introduction

The convolution integral¹ is the fundamental expression relating the input and output of an LTI system. However, it has three shortcomings:

1. It can be tedious to calculate.
2. It offers only limited physical interpretation of what the system is actually doing.
3. It gives little insight on how to design systems to accomplish certain tasks.

The Fourier Series², along with the Fourier Transform and Laplace Transform, provides a way to address these three points. Central to all of these methods is the concept of an eigenfunction³ (or eigenvector⁴). We will look at how we can rewrite any given signal, $f(t)$, in terms of complex exponentials⁵.

In fact, by making our notions of signals and linear systems more mathematically abstract, we will be able to draw enlightening parallels between signals and systems and linear algebra⁶.

2 Eigenfunctions and LTI Systems

The action of a LTI system $\mathcal{H}[\dots]$ on one of its eigenfunctions e^{st} is

1. extremely easy (and fast) to calculate

$$\mathcal{H}[e^{st}] = H[s] e^{st} \tag{1}$$

2. easy to interpret: $\mathcal{H}[\dots]$ just **scales** e^{st} , keeping its frequency constant.

If only every function were an eigenfunction of $\mathcal{H}[\dots]$...

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¹"Continuous Time Convolution" <<http://cnx.org/content/m10085/latest/>>

²"Fourier Series: Eigenfunction Approach" <<http://cnx.org/content/m10496/latest/>>

³"Eigenfunctions of LTI Systems" <<http://cnx.org/content/m10500/latest/>>

⁴"Matrix Diagonalization" <<http://cnx.org/content/m10738/latest/>>

⁵"Continuous Time Complex Exponential" <<http://cnx.org/content/m10060/latest/>>

⁶"Linear Algebra: The Basics" <<http://cnx.org/content/m10734/latest/>>

2.1 LTI System

... of course, not every function can be, but for LTI systems, their eigenfunctions span⁷ the space of periodic functions⁸, meaning that for (almost) any periodic function $f(t)$ we can find $\{c_n\}$ where $n \in \mathbb{Z}$ and $c_i \in \mathbb{C}$ such that:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_0 n t} \tag{2}$$

Given (2), we can rewrite $\mathcal{H}[f(t)] = y(t)$ as the following system

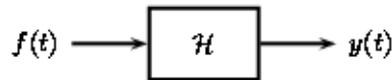


Figure 1: Transfer Functions modeled as LTI System.

where we have:

$$f(t) = \sum_n c_n e^{i\omega_0 n t}$$

$$y(t) = \mathcal{H}[f(t)] = \sum_n c_n H(i(\omega_0, n, t)) e^{i\omega_0 n t}$$

This transformation from $f(t)$ to $y(t)$ can also be illustrated through the process below. Note that each arrow indicates an operation on our signal or coefficients.

$$f(t) \rightarrow \{c_n\} \rightarrow \{c_n H(i\omega_0 n)\} \rightarrow y(t) \tag{3}$$

where the three steps (arrows) in the above illustration represent the following three operations:

1. Transform with analysis (Fourier Coefficient⁹ equation):

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-i\omega_0 n t} dt$$

2. Action of \mathcal{H} on the Fourier series¹⁰ - equals a multiplication by $H(i\omega_0 n)$
3. Translate back to old basis - inverse transform using our synthesis equation from the Fourier series:

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_0 n t}$$

⁷"Linear Algebra: The Basics": Section Span <http://cnx.org/content/m10734/latest/#span_sec>

⁸"Continuous Time Periodic Signals" <<http://cnx.org/content/m10744/latest/>>

⁹"Derivation of Fourier Coefficients Equation" <<http://cnx.org/content/m10733/latest/>>

¹⁰"Fourier Series: Eigenfunction Approach" <<http://cnx.org/content/m10496/latest/>>

3 Physical Interpretation of Fourier Series

The Fourier series $\{c_n\}$ of a signal $f(t)$, defined in (2), also has a very important physical interpretation. Coefficient c_n tells us "how much" of frequency $\omega_0 n$ is in the signal.

Signals that vary slowly over time - **smooth signals** - have large c_n for small n .

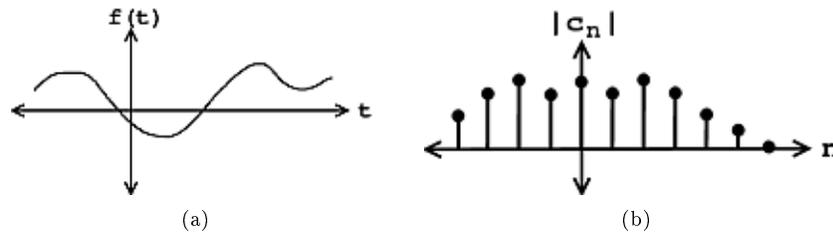


Figure 2: We begin with our smooth signal $f(t)$ on the left, and then use the Fourier series to find our Fourier coefficients - shown in the figure on the right.

Signals that vary quickly with time - **edgy** or **noisy signals** - will have large c_n for large n .

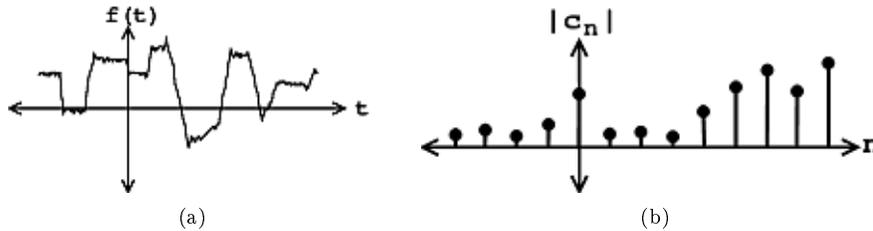


Figure 3: We begin with our noisy signal $f(t)$ on the left, and then use the Fourier series to find our Fourier coefficients - shown in the figure on the right.

Example 1: Periodic Pulse

We have the following pulse function, $f(t)$, over the interval $[-\frac{T}{2}, \frac{T}{2}]$:

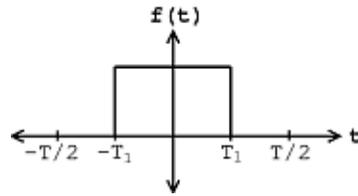


Figure 4: Periodic Signal $f(t)$

Using our formula for the Fourier coefficients,

$$c_n = \frac{1}{T} \int_0^T f(t) e^{-i\omega_0 n t} dt \tag{4}$$

we can easily calculate our c_n . We will leave the calculation as an exercise for you! After solving the the equation for our $f(t)$, you will get the following results:

$$c_n = \begin{cases} \frac{2T_1}{T} & \text{if } n = 0 \\ \frac{2\sin(\omega_0 n T_1)}{n\pi} & \text{if } n \neq 0 \end{cases} \tag{5}$$

For $T_1 = \frac{T}{8}$, see the figure below for our results:

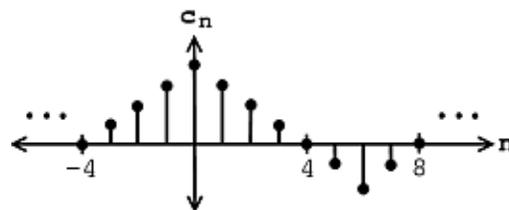


Figure 5: Our Fourier coefficients when $T_1 = \frac{T}{8}$

Our signal $f(t)$ is flat except for two edges (discontinuities). Because of this, c_n around $n = 0$ are large and c_n gets smaller as n approaches infinity.

QUESTION: Why does $c_n = 0$ for $n = \{\dots, -4, 4, 8, 16, \dots\}$? (What part of $e^{-i\omega_0 n t}$ lies over the pulse for these values of n ?)