Orthonormal Basis Expansions[*]

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Abstract

The module looks at decomposing signals through orthonormal basis expansion to provide an alternative representation. The module presents many examples of solving these problems and looks at them in several spaces and dimensions.

1 Main Idea

When working with signals many times it is helpful to break up a signal into smaller, more manageable parts. Hopefully by now you have been exposed to the concept of eigenvectors[1] and there use in decomposing a signal into one of its possible basis. By doing this we are able to simplify our calculations of signals and systems through eigenfunctions of LTI systems[2].

Now we would like to look at an alternative way to represent signals, through the use of **orthonormal basis**. We can think of orthonormal basis as a set of building blocks we use to construct functions. We will build up the signal/vector as a weighted sum of basis elements.

Example 1

The complex sinusoids $\frac{1}{\sqrt{T}}e^{i\omega_0 nt}$ for all $-\infty < n < \infty$ form an orthonormal basis for $L^2([0,T])$.

In our Fourier series[3] equation, $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_0 nt}$, the $\{c_n\}$ are just another representation of $f(t)$.

NOTE: For signals/vectors in a Hilbert Space, the expansion coefficients are easy to find.

2 Alternate Representation

Recall our definition of a **basis**: A set of vectors $\{b_i\}$ in a vector space S is a basis if

1. The b_i are linearly independent.

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[1]"Eigenvectors and Eigenvalues" <http://cnx.org/content/m10736/latest/>

[2]"Eigenfunctions of LTI Systems" <http://cnx.org/content/m10500/latest/>

[3]"Fourier Series: Eigenfunction Approach" <http://cnx.org/content/m10496/latest/>

2. The b_i span^4 S. That is, we can find $\{\alpha_i\}$, where $\alpha_i \in \mathbb{C}$ (scalars) such that

$$\forall x, x \in S : \left(x = \sum_i \alpha_i b_i \right) \tag{1}$$

where x is a vector in S, α is a scalar in $\mathbb{C}$, and b is a vector in S.

Condition 2 in the above definition says we can **decompose** any vector in terms of the $\{b_i\}$. Condition 1 ensures that the decomposition is **unique** (think about this at home).

NOTE: The $\{\alpha_i\}$ provide an alternate representation of x.

Example 2
Let us look at simple example in $\mathbb{R}^2$, where we have the following vector:

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Standard Basis: $\{e_0, e_1\} = \left\{ (1,0)^T, (0,1)^T \right\}$

$$x = e_0 + 2e_1$$

Alternate Basis: $\{h_0, h_1\} = \left\{ (1,1)^T, (1,-1)^T \right\}$

$$x = \frac{3}{2} h_0 + \frac{-1}{2} h_1$$

In general, given a basis $\{b_0, b_1\}$ and a vector $x \in \mathbb{R}^2$, how do we find the α_0 and α_1 such that

$$x = \alpha_0 b_0 + \alpha_1 b_1 \tag{2}$$

3 Finding the Coefficients

Now let us address the question posed above about finding α_i's in general for $\mathbb{R}^2$. We start by rewriting (2) so that we can stack our b_i's as columns in a 2×2 matrix.

$$\begin{pmatrix} x \end{pmatrix} = \alpha_0 \begin{pmatrix} b_0 \end{pmatrix} + \alpha_1 \begin{pmatrix} b_1 \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} \vdots & \vdots \\ b_0 & b_1 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \tag{4}$$

Example 3
Here is a simple example, which shows a little more detail about the above equations.

$$\begin{aligned} \begin{pmatrix} x[0] \\ x[1] \end{pmatrix} &= \alpha_0 \begin{pmatrix} b_0[0] \\ b_0[1] \end{pmatrix} + \alpha_1 \begin{pmatrix} b_1[0] \\ b_1[1] \end{pmatrix} \\ &= \begin{pmatrix} \alpha_0 b_0[0] + \alpha_1 b_1[0] \\ \alpha_0 b_0[1] + \alpha_1 b_1[1] \end{pmatrix} \end{aligned} \tag{5}$$

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^{4}"Linear Algebra: The Basics": Section Span <http://cnx.org/content/m10734/latest/#span_sec>

$$\begin{pmatrix} x\,[0] \\ x\,[1] \end{pmatrix} = \begin{pmatrix} b_0\,[0] & b_1\,[0] \\ b_0\,[1] & b_1\,[1] \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \tag{6}$$

3.1 Simplifying our Equation

To make notation simpler, we define the following two items from the above equations:

- **Basis Matrix**:

$$B = \begin{pmatrix} \vdots & \vdots \\ b_0 & b_1 \\ \vdots & \vdots \end{pmatrix}$$

- **Coefficient Vector**:

$$\alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

This gives us the following, concise equation:

$$x = B\alpha \tag{7}$$

which is equivalent to $x = \sum_{i=0}^{1} \alpha_i b_i$.

Example 4

Given a standard basis, $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, then we have the following basis matrix:

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To get the α_i's, we solve for the coefficient vector in (7)

$$\alpha = B^{-1}x \tag{8}$$

Where B^{-1} is the inverse matrix[5] of B.

3.2 Examples

Example 5

Let us look at the standard basis first and try to calculate α from it.

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

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[5]"Matrix Inversion" $<$http://cnx.org/content/m2113/latest/$>$

Where I is the **identity matrix**. In order to solve for α let us find the inverse of B first (which is obviously very trivial in this case):

$$B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore we get,

$$\alpha = B^{-1}x = x$$

Example 6

Let us look at a ever-so-slightly more complicated basis of $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} = \{h_0, h_1\}$ Then our basis matrix and inverse basis matrix becomes:

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}$$

and for this example it is given that

$$x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Now we solve for α

$$\alpha = B^{-1}x = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 0.5 \end{pmatrix}$$

and we get

$$x = 2.5h_0 + 0.5h_1$$

Exercise 1   *(Solution on p. 6.)*

Now we are given the following basis matrix and x:

$$\{b_0, b_1\} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$$

$$x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

For this problem, make a sketch of the bases and then represent x in terms of b_0 and b_1.

NOTE: A change of basis simply looks at x from a "different perspective." B^{-1}**transforms**x from the standard basis to our new basis, $\{b_0, b_1\}$. Notice that this is a totally mechanical procedure.

4 Extending the Dimension and Space

We can also extend all these ideas past just $\mathbb{R}^2$ and look at them in $\mathbb{R}^n$ and $\mathbb{C}^n$. This procedure extends naturally to higher (> 2) dimensions. Given a basis $\{b_0, b_1, \ldots, b_{n-1}\}$ for $\mathbb{R}^n$, we want to find $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ such that

$$x = \alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_{n-1} b_{n-1} \tag{9}$$

Again, we will set up a basis matrix

$$B = \begin{pmatrix} b_0 & b_1 & b_2 & \ldots & b_{n-1} \end{pmatrix}$$

where the columns equal the basis vectors and it will always be an n×n matrix (although the above matrix does not appear to be square since we left terms in vector notation). We can then proceed to rewrite (7)

$$x = \begin{pmatrix} b_0 & b_1 & \ldots & b_{n-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = B\alpha$$

and

$$\alpha = B^{-1} x$$

Solutions to Exercises in this Module

Solution to Exercise (p. 4)

In order to represent x in terms of b_0 and b_1 we will follow the same steps we used in the above example.

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{-1}{6} \end{pmatrix}$$

$$\alpha = B^{-1}x = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix}$$

And now we can write x in terms of b_0 and b_1.

$$x = b_0 + \frac{2}{3}b_1$$

And we can easily substitute in our known values of b_0 and b_1 to verify our results.