Function Space*

Justin Romberg

Abstract

This module gives an example on function space.

We can also find basis vectors[1] for vector spaces[2] other than $\mathbb{R}^n$.

Let P_n be the vector space of n-th order polynomials on (-1, 1) with real coefficients (verify P_2 is a *v.s.* at home).

Example 1

$P_2 = \{\text{all quadratic polynomials}\}$. Let $b_0(t) = 1$, $b_1(t) = t$, $b_2(t) = t^2$.

$\{b_0(t), b_1(t), b_2(t)\}$ **span** P_2, *i.e.* you can write any $f(t) \in P_2$ as

$$f(t) = \alpha_0 b_0(t) + \alpha_1 b_1(t) + \alpha_2 b_2(t)$$

for some $\alpha_i \in \mathbb{R}$.

NOTE: P_2 is 3 dimensional.

$$f(t) = t^2 - 3t - 4$$

Alternate basis

$$\{b_0(t), b_1(t), b_2(t)\} = \left\{1, t, \frac{1}{2}\left(3t^2 - 1\right)\right\}$$

write $f(t)$ in terms of this new basis $d_0(t) = b_0(t)$, $d_1(t) = b_1(t)$, $d_2(t) = \frac{3}{2}b_2(t) - \frac{1}{2}b_0(t)$.

$$f(t) = t^2 - 3t - 4 = 4b_0(t) - 3b_1(t) + b_2(t)$$

$$f(t) = \beta_0 d_0(t) + \beta_1 d_1(t) + \beta_2 d_2(t) = \beta_0 b_0(t) + \beta_1 b_1(t) + \beta_2\left(\frac{3}{2}b_2(t) - \frac{1}{2}b_0(t)\right)$$

$$f(t) = \left(\beta_0 - \frac{1}{2}\right) b_0(t) + \beta_1 b_1(t) + \frac{3}{2}\beta_2 b_2(t)$$

so

$$\beta_0 - \frac{1}{2} = 4$$

*Version 2.5: Oct 1, 2009 10:26 pm GMT-5

†http://creativecommons.org/licenses/by/1.0

[1] "Orthonormal Basis Expansions" <http://cnx.org/content/m10760/latest/>

[2] "Vector Spaces" <http://cnx.org/content/m10767/latest/>

$$\beta_1 = -3$$

$$\frac{3}{2}\beta_2 = 1$$

then we get

$$f(t) = 4.5d_0(t) - 3d_1(t) + \frac{2}{3}d_2(t)$$

Example 2

$e^{i\omega_0 nt}|_{n=-\infty}^{\infty}$ is a basis for $L^2([0,T])$, $T = \frac{2\pi}{\omega_0}$, $f(t) = \sum_n \left(C_n e^{i\omega_0 nt}\right)$.

We calculate the expansion coefficients with

"change of basis" formula

$$C_n = \frac{1}{T}\int_0^T f(t)\, e^{-(i\omega_0 nt)} dt \quad (1)$$

NOTE: There are an infinite number of elements in the basis set, that means $L^2([0,T])$ is infinite dimensional (scary!).

Infinite-dimensional spaces are hard to visualize. We can get a handle on the intuition by recognizing they share many of the same mathematical properties with finite dimensional spaces. Many concepts apply to both (like "basis expansion"). Some don't (change of basis isn't a nice matrix formula).