M/M/1 Arrival and Departure Time Occupancy Distributions*

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The continuous-time chain for the M/M/1 queue has the following state transition graph.

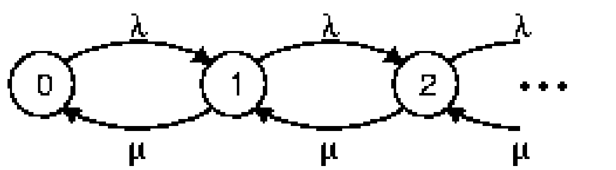

Figure 1

The state probabilities are given by $\pi_k = \rho^k (1 - \rho)$.
Looking at the M/M/1 system only at transistions gives rise to the following discrete-time chain.

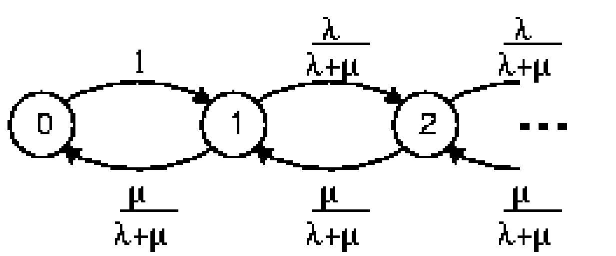

Figure 2

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The single-step probability matrix for the discrete-time chain is

$$P=\left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & \dots \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & 0 & \dots \\ 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \dots \\ 0 & 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}\right) \quad (1)$$

Solving $\pi' P=\pi'$ yields

$$\left({\pi'}_1\left(\frac{\mu}{\lambda+\mu}\right)={\pi'}_0\right)\Rightarrow\left({\pi'}_1=(1+\rho)\,{\pi'}_0\right)$$

$$\left({\pi'}_0\,(1)+{\pi'}_2\left(\frac{\mu}{\lambda+\mu}\right)={\pi'}_1\right)\Rightarrow\left({\pi'}_2=(1+\rho)\,\rho{\pi'}_0\right)$$

$$\left({\pi'}_1\left(\frac{\lambda}{\lambda+\mu}\right)+{\pi'}_3\left(\frac{\mu}{\lambda+\mu}\right)={\pi'}_2\right)\Rightarrow\left({\pi'}_3=(1+\rho)\,\rho^2{\pi'}_0\right)$$

$$\left({\pi'}_2\left(\frac{\lambda}{\lambda+\mu}\right)+{\pi'}_4\left(\frac{\mu}{\lambda+\mu}\right)={\pi'}_3\right)\Rightarrow\left({\pi'}_4=(1+\rho)\,\rho^3{\pi'}_0\right)$$

and in general, $\forall k, k\geq 1:\left({\pi'}_k=(1+\rho)\,\rho^{k-1}{\pi'}_0\right)$.

Normalization:

$$\sum_{k=0}^{\infty}{\pi'}_k={\pi'}_0+\sum_{k=1}^{\infty}(1+\rho)\,\rho^{k-1}{\pi'}_0={\pi'}_0\left(1+(1+\rho)\sum_{i=0}^{\infty}\rho^i\right)={\pi'}_0\left(1+\frac{1+\rho}{1-\rho}\right)=1$$

$${\pi'}_0=\frac{1-\rho}{2}$$

$${\pi'}_{k\geq 1}=\frac{\rho^{k-1}\left(1-\rho^2\right)}{2}$$

$$\begin{array}{rcl} Pr\,[\text{arrival sees k jobs ahead of it}] & = & Pr\,[\text{arrival in state k }\mid\text{arrival}] \\ & = & \frac{Pr[\text{arrival in state k \& arrival}]}{Pr[\text{arrival}]} \\ & = & \frac{Pr[\text{arrival in state k}]}{Pr[\text{arrival}]} \end{array} \quad (2)$$

$$\begin{array}{rcl} Pr\,[\text{arrival}] & = & \sum_{k=0}^{\infty}{\pi'}_k Pr\,[\text{arrival }\mid\text{state k}] \\ & = & {\pi'}_0 1+\sum_{k=1}^{\infty}\frac{\rho^{k-1}\left(1-\rho^2\right)}{2}\frac{\lambda}{\lambda+\mu} \\ & = & \frac{1-\rho}{2}+\sum_{k=1}^{\infty}\frac{\rho^{k-1}(1-\rho)}{2}\frac{\lambda+\mu}{\lambda}\frac{\lambda}{\lambda+\mu} \\ & = & \frac{1-\rho}{2}+\frac{1-\rho}{2}\sum_{k=1}^{\infty}\rho^k \\ & = & \frac{1-\rho}{2}+\frac{1-\rho}{2}\left(\frac{1}{1-\rho}-1\right) \\ & = & \frac{1}{2} \end{array} \quad (3)$$

which could have been inferred directly since, for the system to be stable, half of the transistions must be arrivals and half must be departures (in the long run).

$$Pr\,[\text{arrival sees k jobs ahead of it}]=\frac{{\pi'}_k\frac{\lambda}{\lambda+\mu}}{\frac{1}{2}}=\rho^k\,(1-\rho)$$

What about departing jobs?

$$
\begin{aligned}
Pr\left[\text{departing job leaves k jobs}\right] &= Pr\left[\text{departure in state k+1} \mid \text{departure}\right] \\
&= \frac{Pr[\text{departure in state k+1}]}{Pr[\text{departure}]} \\
&= \frac{\frac{\rho^k\left(1-\rho^2\right)}{2}\frac{\mu}{\lambda+\mu}}{\frac{1}{2}} \\
&= \frac{\frac{\rho^k(1-\rho)}{2}\frac{\lambda+\mu}{\mu}\frac{\mu}{\lambda+\mu}}{\frac{1}{2}} \\
&= \rho^k\left(1-\rho\right)
\end{aligned}
\tag{4}
$$

Hence, the probability that an arrival sees k jobs ahead of it is equal to the probability that a departure leaves k jobs behind it, which is equal to the probability that the system is in state k in steady-state.