## HILBERT SPACES\*

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## Abstract

This module will provide an introduction to the concepts of Hilbert spaces.

## **1** Hilbert Spaces

A vector space S with a valid inner product<sup>1</sup> defined on it is called an **inner product space**, which is also a **normed linear space**. A **Hilbert space** is an inner product space that is complete with respect to the norm defined using the inner product. Hilbert spaces are named after David Hilbert<sup>2</sup>, who developed this idea through his studies of integral equations. We define our valid norm using the inner product as:

$$\|x\| = \sqrt{\langle x, x \rangle} \tag{1}$$

Hilbert spaces are useful in studying and generalizing the concepts of Fourier expansion, Fourier transforms, and are very important to the study of quantum mechanics. Hilbert spaces are studied under the functional analysis branch of mathematics.

## 1.1 Examples of Hilbert Spaces

Below we will list a few examples of Hilbert spaces<sup>3</sup>. You can verify that these are valid inner products at home.

• For  $\mathbb{C}^n$ ,

$$\langle x, y \rangle = y^T x = \begin{pmatrix} \overline{y_0} & \overline{y_1} & \dots & \overline{y_{n-1}} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \sum_{i=0}^{n-1} x_i \overline{y_i}$$

• Space of finite energy complex functions:  $L^{2}(\mathbb{R})$ 

$$\langle f,g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt$$

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<sup>&</sup>lt;sup>1</sup>"Inner Products" < http://cnx.org/content/m10755/latest/>

<sup>&</sup>lt;sup>2</sup>http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hilbert.html

 $<sup>^{3}</sup>$ "Hilbert Spaces" <http://cnx.org/content/m10434/latest/>

• Space of square-summable sequences:  $\ell^2(\mathbb{Z})$ 

$$< x, y > = \sum_{i=-\infty}^{\infty} x \left[i\right] \overline{y \left[i\right]}$$