Wavelet Dilation Equation*

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1 A Condition for Existence

Dilation Equation

$$\phi(t) = \sqrt{2}\sum_{n}(h(n)\,\phi(2t-n)) \tag{1}$$

$$\int \phi(t)\,dt = \int \sqrt{2}\sum_{n}(h(n)\,\phi(2t-n))\,dt \tag{2}$$

$$\int \phi(t)\,dt = \sqrt{2}\sum_{n}\left(h(n)\int \phi(2t-n)\,dt\right) \tag{3}$$

$$\int \phi(t)\,dt = \sqrt{2}\sum_{n}\left(h(n)\int \phi(2t)\,dt\right) \tag{4}$$

$$\int \phi(t)\,dt = \sqrt{2}\left(\sum_{n}(h(n))\right)\int \phi(2t)\,dt \tag{5}$$

$$\int \phi(t)\,dt = \sqrt{2}\left(\sum_{n}(h(n))\right)\frac{1}{2}\int \phi(t)\,dt \tag{6}$$

$$1 = \frac{1}{\sqrt{2}}\sum_{n}(h(n)) \tag{7}$$

assuming $\int \phi(t)\,dt \neq 0$. For a solution $\phi(t)$ to even exist, it is required that the coefficients $h(n)$ add up to $\sqrt{2}$.

$$\sum_{n}(h(n)) = \sqrt{2} \tag{8}$$

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2 Fourier Form of the Dilation Equation

Dilation Equation

$$\phi(t) = \sqrt{2}\sum_n (h(n)\phi(2t-n)) \tag{9}$$

$$\mathcal{F}\{\mathrm{LHS}\} = \mathcal{F}\{\mathrm{RHS}\}$$

$$\mathcal{F}\{\phi(t)\} = \mathcal{F}\left\{\sqrt{2}\sum_n (h(n)\phi(2t-n))\right\}$$

$$\Phi(\omega) = \sqrt{2}\sum_n (h(n)\mathcal{F}\{\phi(2t-n)\})$$

$$\Phi(\omega) = \sqrt{2}\sum_n \left(h(n)\int_{-\infty}^{\infty}\phi(2t-n)e^{-(i\omega t)}dt\right)$$

For $\tau = 2t-n$,

$$\Phi(\omega) = \sqrt{2}\sum_n \left(h(n)\frac{1}{2}\int_{-\infty}^{\infty}\phi(\tau)e^{-\left(i\omega\frac{\tau+n}{2}\right)}d\tau\right)$$

$$\Phi(\omega) = \frac{1}{\sqrt{2}}\sum_n \left(h(n)e^{-\left(i\frac{\omega}{2}n\right)}\int_{-\infty}^{\infty}\phi(\tau)e^{-\left(i\frac{\omega}{2}\tau\right)}d\tau\right)$$

$$\Phi(\omega) = \frac{1}{\sqrt{2}}\sum_n \left(h(n)e^{-\left(i\frac{\omega}{2}n\right)}\Phi\left(\frac{\omega}{2}\right)\right)$$

$$\Phi(\omega) = \frac{1}{\sqrt{2}}\Phi\left(\frac{\omega}{2}\right)\sum_n \left(h(n)e^{-\left(i\frac{\omega}{2}n\right)}\right)$$

Note that

$$H^f(\omega) = \mathrm{DTFT}(h(n)) = \sum_n \left(h(n)e^{-(i\omega n)}\right)$$

is the discrete-time Fourier transform (DTFT) of the scaling filter $h(n)$. We then have the Fourier form of the dilation equation:

$$\Phi(\omega) = \frac{1}{\sqrt{2}}H^f\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right) \tag{10}$$

When $\omega = 0$ is put into this equation, we get

$$\Phi(0) = \frac{1}{\sqrt{2}}H^f(0)\Phi(0)$$

or

$$H^f(0) = \sqrt{2} \tag{11}$$

which is exactly the same equation as $\sum_n (h(n)) = \sqrt{2}$ that we already got.

From (10), we can write

$$\Phi\left(\frac{\omega}{2}\right) = \frac{1}{\sqrt{2}}H^f\left(\frac{\omega}{4}\right)\Phi\left(\frac{\omega}{4}\right)$$

or

$$\Phi(\omega) = \frac{1}{\sqrt{2}}H^f\left(\frac{\omega}{2}\right)\frac{1}{\sqrt{2}}H^f\left(\frac{\omega}{4}\right)\Phi\left(\frac{\omega}{4}\right)$$

If we iterate, we get the **infinite-product** formula for $\Phi(\omega)$,

$$\Phi(\omega) = \Phi(0) \prod_{k=1}^{\infty} \left(\frac{1}{\sqrt{2}} H^f \left(\frac{\omega}{2^k} \right) \right) \tag{12}$$

From this formula, we can see that the zeros of $\Phi(\omega)$ are determined by the zeros of $H^f(\omega)$.

3 What is the support of ϕ(t)?

Given $h(n)$, what is the support of $\phi(t)$?

Suppose $h(n)$ is supported on $0 \leq n \leq N-1$ then from the dilation equation

$$\phi(t) = \sqrt{2} \sum_{n=0}^{N-1} (h(n)\phi(2t-n))$$

the support of $\phi(t)$ is found by matching the support of the left hand side and the right hand side.

Suppose the support of $\phi(t)$ is $[a,b]$. Then

- $\phi(2t)$ has support $\left[\frac{a}{2}, \frac{b}{2}\right]$
- $\phi(2t-1)$ has support $\left[\frac{a+1}{2}, \frac{b+1}{2}\right]$
- $\vdots$
- $\phi(2t-k)$ has support $\left[\frac{a+k}{2}, \frac{b+k}{2}\right]$
- $\vdots$
- $\phi(2t-(N-1))$ has support $\left[\frac{a+N-1}{2}, \frac{b+N-1}{2}\right]$

The support of the LHS is $[a,b]$ by assumption. The support of the RHS is $\left[\frac{a}{2}, \frac{b}{2} + \frac{N-1}{2}\right]$. Therefore, matching the endpoints we get

$$a = \frac{a}{2}$$

$$b = \frac{b}{2} + \frac{N-1}{2}$$

which gives

$$a = 0$$

$$b = N-1$$

The support of $\phi(t)$ is $[0, N-1]$.

$$h(n) \quad \text{has finite support} \quad \Rightarrow \phi(t) \quad \text{of finite support}$$

That means that there are **finite** degrees of freedom in design of finitely supported scaling functions.

4 Sum-Integral Equality of ϕ(t)

Note:

$$\lim_{j \to \infty} \frac{1}{2^j} \sum_{l} \left(\phi \left(\frac{l}{2^j} \right) \right) = \int \phi(t)\, dt$$

Let us define

$$\forall j, j \geq 1 : \left(\left(S_j := \frac{1}{2^j} \sum_{l} \left(\phi \left(\frac{l}{2^j} \right) \right) \right) \right)$$

Begin with the dilation equation

$$\phi(t)=\sqrt{2}\sum_n\left(h(n)\,\phi(2t-n)\right)$$

and substitute $\frac{l}{2^j}$ for t.

$$\phi\left(\frac{l}{2^j}\right)=\sqrt{2}\sum_n\left(h(n)\,\phi\left(\frac{l}{2^{j-1}}-n\right)\right)$$

with $j\geq 1$. Now sum over l:

$$\sum_l\left(\phi\left(\frac{l}{2^j}\right)\right)=\sum_l\left(\sqrt{2}\sum_n\left(h(n)\,\phi\left(\frac{l}{2^{j-1}}-n\right)\right)\right)$$

$$\sum_l\left(\phi\left(\frac{l}{2^j}\right)\right)=\sqrt{2}\sum_n\left(h(n)\sum_l\left(\phi\left(\frac{l}{2^{j-1}}-n\right)\right)\right)$$

$$\sum_l\left(\phi\left(\frac{l}{2^j}\right)\right)=\sqrt{2}\sum_n\left(h(n)\sum_l\left(\phi\left(\frac{l}{2^{j-1}}\right)\right)\right)$$

$$\sum_l\left(\phi\left(\frac{l}{2^j}\right)\right)=2\sum_l\left(\phi\left(\frac{l}{2^{j-1}}\right)\right)$$

Divide both sides by 2^j:

$$\frac{1}{2^j}\sum_l\left(\phi\left(\frac{l}{2^j}\right)\right)=\frac{1}{2^{j-1}}\sum_l\left(\phi\left(\frac{l}{2^{j-1}}\right)\right)$$

$$S_j=S_{j-1}$$

For $j\geq 1$, all the S_j are the same!

$$\underset{j\to\infty}{lim}S_j=S_0$$

Therefore

$$\int\phi(t)\,dt=\sum_k\left(\phi(k)\right)\tag{13}$$

If $\phi(t)$ is continuous, then it can also be shown that

$$\int\phi(t)\,dt=\sum_k\left(\phi(t_o+k)\right)\tag{14}$$

for any real t_o.