Images: 2D signals*

Robert Nowak

Abstract

This module introduces image processing, 2D convolution, 2D sampling and 2D FTs.

1 Image Processing

Image not finished

Figure 1: Images are 2D functions $f(x,y)$

2 Linear Shift Invariant Systems

Image not finished

Figure 2

H is LSI if:

1. $$H(\alpha_1 f_1(x,y) + \alpha_2 f_2(x,y)) = H(f_1(x,y)) + H(f_2(x,y))$$
   for all images f_1, f_2 and scalar.
2. $$H(f(x-x_0, y-y_0)) = g(x-x_0, y-y_0)$$

*Version 2.7: Jul 19, 2005 8:11 pm GMT-5

LSI systems are expressed mathematically as 2D convolutions:

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\alpha, y-\beta)\, f(\alpha,\beta)\, d\alpha d\beta$$

where $h(x,y)$ is the 2D impulse response (also called the **point spread function**).

3 2D Fourier Analysis

$$\mathcal{F}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-(iux)} e^{-(ivy)} dx dy$$

where $\mathcal{F}$ is the 2D FT and u and v are frequency variables in $x\,(u)$ and $y\,(v)$.

2D complex exponentials are eigenfunctions[1] for 2D LSI systems:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\alpha, y-\beta)\, e^{iu_0\alpha} e^{iv_0\beta} d\alpha d\beta &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha',\beta')\, e^{iu_0(x-\alpha')} e^{iv_0(y-\beta')} d\alpha\prime d\beta\prime \\ &= e^{iu_0x} e^{iv_0y} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha',\beta')\, e^{-(iu_0\alpha')} e^{-(iv_0\beta')} d\alpha\prime d\beta\prime\end{aligned} \quad (1)$$

where

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha',\beta')\, e^{-(iu_0\alpha')} e^{-(iv_0\beta')} d\alpha\prime d\beta\prime \equiv H(u_0, v_0)$$

$H(u_0, v_0)$ is the 2D Fourier transform of $h(x,y)$ evaluated at frequencies u_0 and v_0.

Figure 3

$$\begin{aligned} g(x,y) &= h(x,y) * f(x,y) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\alpha, y-\beta)\, f(\alpha,\beta)\, d\alpha d\beta \end{aligned} \quad (2)$$

$$G(u,v) = H(u,v)\, \mathcal{F}(u,v)$$

Inverse 2D FT

$$g(x,y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u,v)\, e^{iux} e^{ivy} du dv \quad (3)$$

4 2D Sampling Theory

Figure 4: Think of the image as a 2D surface.

[1] "Eigenfunctions of LTI Systems" <http://cnx.org/content/m10500/latest/>

We can **sample** the height of the surface using a 2D impulse array.

Image not finished

Figure 5: Impulses spaced $\Delta(x)$ apart in the horizontal direction and $\Delta(y)$ in the vertical

$$f_s(x,y) = S(x,y)\,f(x,y)$$

where $f_s(x,y)$ is sampled image in frequency
2D FT of $s(x,y)$ is a 2D impulse array in frequency $S(u,v)$

Image not finished

Figure 6

$$\text{multiplication in time} \Leftrightarrow \text{convolution in frequency}$$

$$F_s(u,v) = S(u,v) * \mathcal{F}(u,v)$$

Image not finished

Figure 7: $\mathcal{F}(u,v)$ is bandlimited in both the horizontal and vertical directions.

Image not finished

Figure 8: periodically replicated in (u,v) frequency plane

5 Nyquist Theorem

Assume that $f(x,y)$ is bandlimited to $\pm(B_x)$, $\pm(B_y)$:

Image not finished

Figure 9

If we sample $f(x,y)$ at spacings of $\Delta(x) < \frac{\pi}{B_x}$ and $\Delta(y) < \frac{\pi}{B_y}$, then $f(x,y)$ can be perfectly recovered from the samples by lowpass filtering:

Image not finished

Figure 10: ideal lowpass filter, 1 inside rectangle, 0 outside

Aliasing in 2D

Image not finished

(a)

Image not finished

(b)

Figure 11