Fast Convolution Using the FFT*

Robert Nowak

This work is produced by OpenStax-CNX and licensed under the
Creative Commons Attribution License 1.0†

Abstract

This module describes FFT, convolution, filtering, LTI systems, digital filters and circular convolution.

1 Important Application of the FFT

Exercise 1 ***(Solution on p. 4.)***

How many complex multiplies and adds are required to convolve two N-pt sequences?

$$y[n] = \sum_{m=0}^{N-1} x[m]\, h[n-m]$$

1. Zero-pad these two sequences to length $2N-1$, take DFTs using the FFT algorithm

   $$x[n] \to X[k]$$

   $$h[n] \to H[k]$$

   The cost is

   $$O((2N-1)\log(2N-1)) = O(N\log N)$$

2. Multiply DFTs

   $$X[k]\, H[k]$$

   The cost is

   $$O(2N-1) = O(N)$$

3. Inverse DFT using FFT

   $$X[k]\, H[k] \to y[n]$$

   The cost is

   $$O((2N-1)\log(2N-1)) = O(N\log N)$$

*Version 2.6: Nov 2, 2005 9:36 am -0600

†http://creativecommons.org/licenses/by/1.0

So the total cost for direct convolution of two N-point sequences is $O\left(N^2\right)$. Total cost for convolution using FFT algorithm is $O\left(N\log N\right)$. That is a **huge savings** (Figure 1).

Image not finished

Figure 1

2 Summary of DFT

- $x\left[n\right]$ is an N-point signal (Figure 2).

Image not finished

Figure 2

- $$X\left[k\right]=\sum_{n=0}^{N-1}x\left[n\right]e^{-\left(i\frac{2\pi}{N}kn\right)}=\sum_{n=0}^{N-1}x\left[n\right]{W_N}^{kn}$$

  where $W_N=e^{-\left(i\frac{2\pi}{N}\right)}$ is a "twiddle factor" and the first part is the basic DFT.

2.1 What is the DFT

$$X\left[k\right]=X\left[F=\frac{k}{N}\right]=\sum_{n=0}^{N-1}x\left[n\right]e^{-\left(i2\pi Fn\right)}$$

where $X\left[F=\frac{k}{N}\right]$ is the DTFT of $x\left[n\right]$ and $\sum_{n=0}^{N-1}x\left[n\right]e^{-\left(i2\pi Fn\right)}$ is the DTFT of $x\left[n\right]$ at digital frequency F. This is a sample of the DTFT. We can do frequency domain analysis on a computer!

2.2 Inverse DFT (IDFT)

$$x\left[n\right]=\frac{1}{N}\sum_{n=0}^{N-1}X\left[k\right]e^{i\frac{2\pi}{N}kn}$$

- Build $x\left[n\right]$ using **Simple** complex sinusoidal **building block** signals
- Amplitude of each complex sinusoidal building block in $x\left[n\right]$ is $\frac{1}{N}X\left[k\right]$

2.3 Circular Convolution

DFT

$$x[n] \oplus h[n] \leftrightarrow X[k]\,H[k] \tag{1}$$

2.4 Regular Convolution from Circular Convolution

- Zero pad $x[n]$ and $h[n]$ to length $= \text{length}(x) + \text{length}(h) - 1$
- Zero padding increases frequency resolution in DFT domain (Figure 3)

Image not finished (a) Image not finished (b)

Figure 3: (a) 8-pt DFT of 8-pt signal (b) 16-pt DFT of same signal padded with 8 additional zeros

---

2.5 The Fast Fourier Transform (FFT)

- Efficient computational algorithm for calculating the DFT
- "Divide and conquer"
- Break signal into even and odd samples keep taking shorter and shorter DFTs, then build N-pt DFT by cleverly combining shorter DFTs
- N-pt DFT: $O\left(N^2\right) \rightarrow O\left(N\log_2 N\right)$

2.6 Fast Convolution

- Use FFT's to compute circular convolution of zero-padded signals
- Much faster than regular convolution if signal lengths are long
- $O\left(N^2\right) \rightarrow O\left(N\log_2 N\right)$

See Figure 4.

Image not finished

Figure 4

Solutions to Exercises in this Module

Solution to Exercise (p. 1)

There are $2N-1$ non-zero output points and each will be computed using N complex mults and $N-1$ complex adds. Therefore,

$$\text{Total Cost} = (2N-1)(N+N-1) \simeq O\left(N^2\right)$$