The DFT: Frequency Domain with a Computer Analysis*

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Abstract

This module will take the ideas of sampling CT signals further by examining how such operations can be performed in the frequency domain and by using a computer.

1 Introduction

We just covered ideal (and non-ideal) (time) sampling of CT signals[1]. This enabled DT signal processing solutions for CT applications (Figure 1):

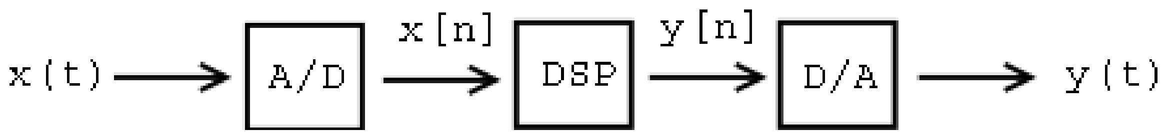

Figure 1

Much of the theoretical analysis of such systems relied on frequency domain representations. How do we carry out these frequency domain analysis on the computer? Recall the following relationships:

$$x[n] \overset{\text{DTFT}}{\leftrightarrow} X(\omega)$$

$$x(t) \overset{\text{CTFT}}{\leftrightarrow} X(\Omega)$$

where ω and Ω are continuous frequency variables.

*Version 2.3: Jul 22, 2005 3:08 pm -0500

[1]"Sampling CT Signals: A Frequency Domain Perspective" <http://cnx.org/content/m10994/latest/>

1.1 Sampling DTFT

Consider the DTFT of a discrete-time (DT) signal $x[n]$. Assume $x[n]$ is of finite duration N (*i.e.*, an N-point signal).

$$X(\omega) = \sum_{n=0}^{N-1} x[n] e^{(-i)\omega n} \tag{1}$$

where $X(\omega)$ is the continuous function that is indexed by the real-valued parameter $-\pi \leq \omega \leq \pi$. The other function, $x[n]$, is a discrete function that is indexed by integers.

We want to work with $X(\omega)$ on a computer. Why not just **sample**$X(\omega)$?

$$\begin{aligned} X[k] &= X\left(\frac{2\pi}{N}k\right) \\ &= \sum_{n=0}^{N-1} x[n] e^{(-i)2\pi\frac{k}{N}n} \end{aligned} \tag{2}$$

In (2) we sampled at $\omega = \frac{2\pi}{N}k$ where $k = \{0, 1, \dots, N-1\}$ and $X[k]$ for $k = \{0, \dots, N-1\}$ is called the **Discrete Fourier Transform (DFT)** of $x[n]$.

Example 1

Finite Duration DT Signal

Image not finished

Figure 2

The DTFT of the image in Figure 2 (Finite Duration DT Signal) is written as follows:

$$X(\omega) = \sum_{n=0}^{N-1} x[n] e^{(-i)\omega n} \tag{3}$$

where ω is any 2π-interval, for example $-\pi \leq \omega \leq \pi$.

Sample X(ω)

Image not finished

Figure 3

where again we sampled at $\omega = \frac{2\pi}{N}k$ where $k = \{0, 1, \dots, M-1\}$. For example, we take

$$M = 10$$

. In the following section (Section 1.1.1: Choosing M) we will discuss in more detail how we should choose M, the number of samples in the 2π interval.

(This is precisely how we would plot $X(\omega)$ in Matlab.)

1.1.1 Choosing M

1.1.1.1 Case 1

Given N (length of $x[n]$), choose $M \gg N$ to obtain a dense sampling of the DTFT (Figure 4):

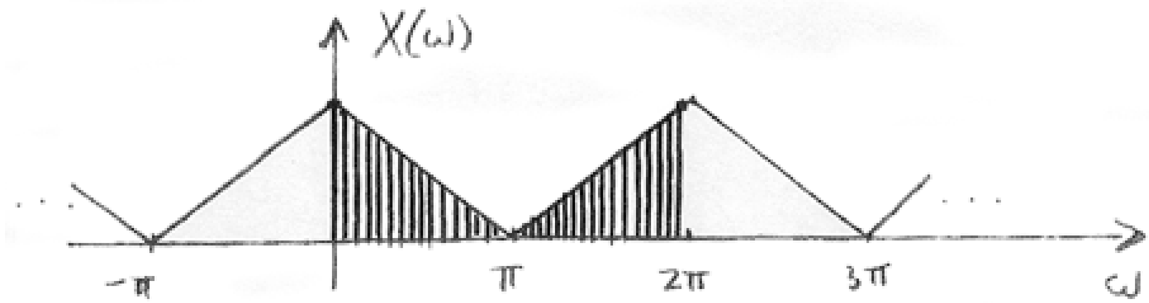

Figure 4

1.1.1.2 Case 2

Choose M as small as possible (to minimize the amount of computation).

In general, we require $M \geq N$ in order to represent all information in

$$\forall n, n = \{0, \dots, N-1\} : (x[n])$$

Let's concentrate on $M = N$:

$$x[n] \overset{\text{DFT}}{\leftrightarrow} X[k]$$

for $n = \{0, \dots, N-1\}$ and $k = \{0, \dots, N-1\}$

$$\text{numbers} \leftrightarrow \text{N} \quad \text{numbers}$$

2 Discrete Fourier Transform (DFT)

Define

$$X[k] \equiv X\left(\frac{2\pi k}{N}\right) \tag{4}$$

where $N = \text{length}(x[n])$ and $k = \{0, \dots, N-1\}$. In this case, $M = N$.

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{(-i)2\pi \frac{k}{N} n} \tag{5}$$

Inverse DFT (IDFT)

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{i2\pi \frac{k}{N} n} \tag{6}$$

2.1 Interpretation

Represent $x[n]$ in terms of a sum of Ncomplex sinusoids[2] of amplitudes $X[k]$ and frequencies

$$\forall k, k \in \{0, \dots, N-1\} : \left(\omega_k = \frac{2\pi k}{N}\right)$$

NOTE: Fourier Series with fundamental frequency $\frac{2\pi}{N}$

2.1.1 Remark 1

IDFT treats $x[n]$ as though it were N-periodic.

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{i2\pi\frac{k}{N}n} \tag{7}$$

where $n \in \{0, \dots, N-1\}$

Exercise 1 *(Solution on p. 11.)*

What about other values of n?

2.1.2 Remark 2

Proof that the IDFT inverts the DFT for $n \in \{0, \dots, N-1\}$

$$\begin{aligned} \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{i2\pi\frac{k}{N}n} &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} x[m] e^{(-i)2\pi\frac{k}{N}m} e^{i2\pi\frac{k}{N}n} \\ &= ??? \end{aligned} \tag{8}$$

Example 2: Computing DFT

Given the following discrete-time signal (Figure 5) with $N = 4$, we will compute the DFT using two different methods (the DFT Formula and Sample DTFT):

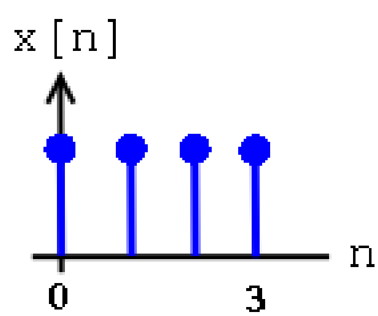

Figure 5

[2]"Continuous Time Complex Exponential" <http://cnx.org/content/m10060/latest/>

1. DFT Formula

$$\begin{array}{rcl} X[k] & = & \sum_{n=0}^{N-1} x[n] e^{(-i)2\pi \frac{k}{N} n} \\ & = & 1 + e^{(-i)2\pi \frac{k}{4}} + e^{(-i)2\pi \frac{k}{4} 2} + e^{(-i)2\pi \frac{k}{4} 3} \\ & = & 1 + e^{(-i)\frac{\pi}{2} k} + e^{(-i)\pi k} + e^{(-i)\frac{3}{2}\pi k} \end{array} \quad (9)$$

Using the above equation, we can solve and get the following results:

$$x[0] = 4$$

$$x[1] = 0$$

$$x[2] = 0$$

$$x[3] = 0$$

2. Sample DTFT. Using the same figure, Figure 5, we will take the DTFT of the signal and get the following equations:

$$\begin{array}{rcl} X(\omega) & = & \sum_{n=0}^{3} e^{(-i)\omega n} \\ & = & \frac{1-e^{(-i)4\omega}}{1-e^{(-i)\omega}} \\ & = & ??? \end{array} \quad (10)$$

Our sample points will be:

$$\omega_k = \frac{2\pi k}{4} = \frac{\pi}{2} k$$

where $k = \{0, 1, 2, 3\}$ (Figure 6).

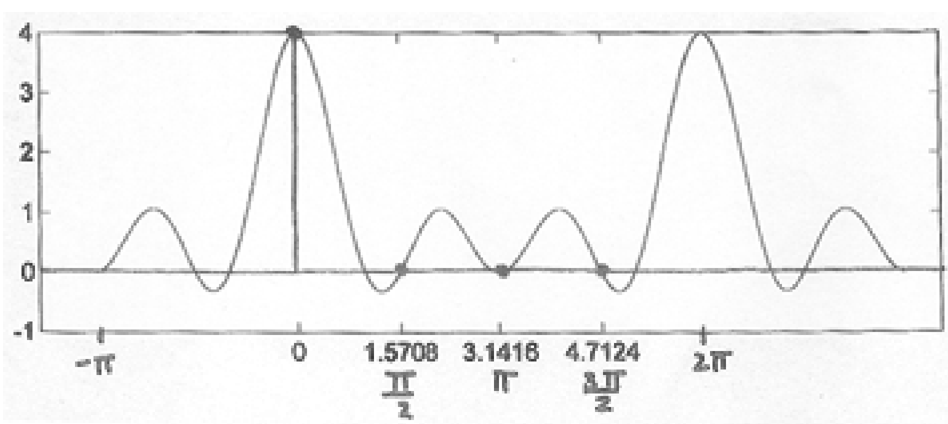

Figure 6

3 Periodicity of the DFT

DFT $X[k]$ consists of **samples** of DTFT, so $X(\omega)$, a 2π-periodic DTFT signal, can be converted to $X[k]$, an N-periodic DFT.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{(-i)2\pi \frac{k}{N} n} \qquad (11)$$

where $e^{(-i)2\pi \frac{k}{N} n}$ is an N-periodic basis function (See Figure 7).

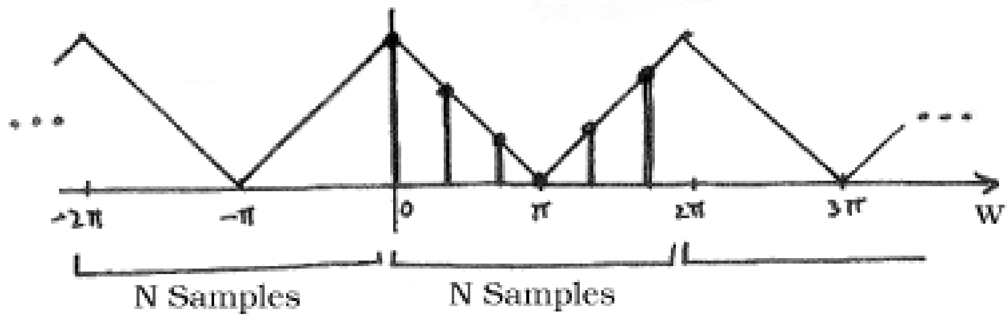

Figure 7

Also, recall,

$$\begin{aligned} x[n] &= \frac{1}{N}\sum_{n=0}^{N-1} X[k]\, e^{i2\pi \frac{k}{N} n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} X[k]\, e^{i2\pi \frac{k}{N} (n+mN)} \\ &= ??? \end{aligned} \qquad (12)$$

Example 3: Illustration

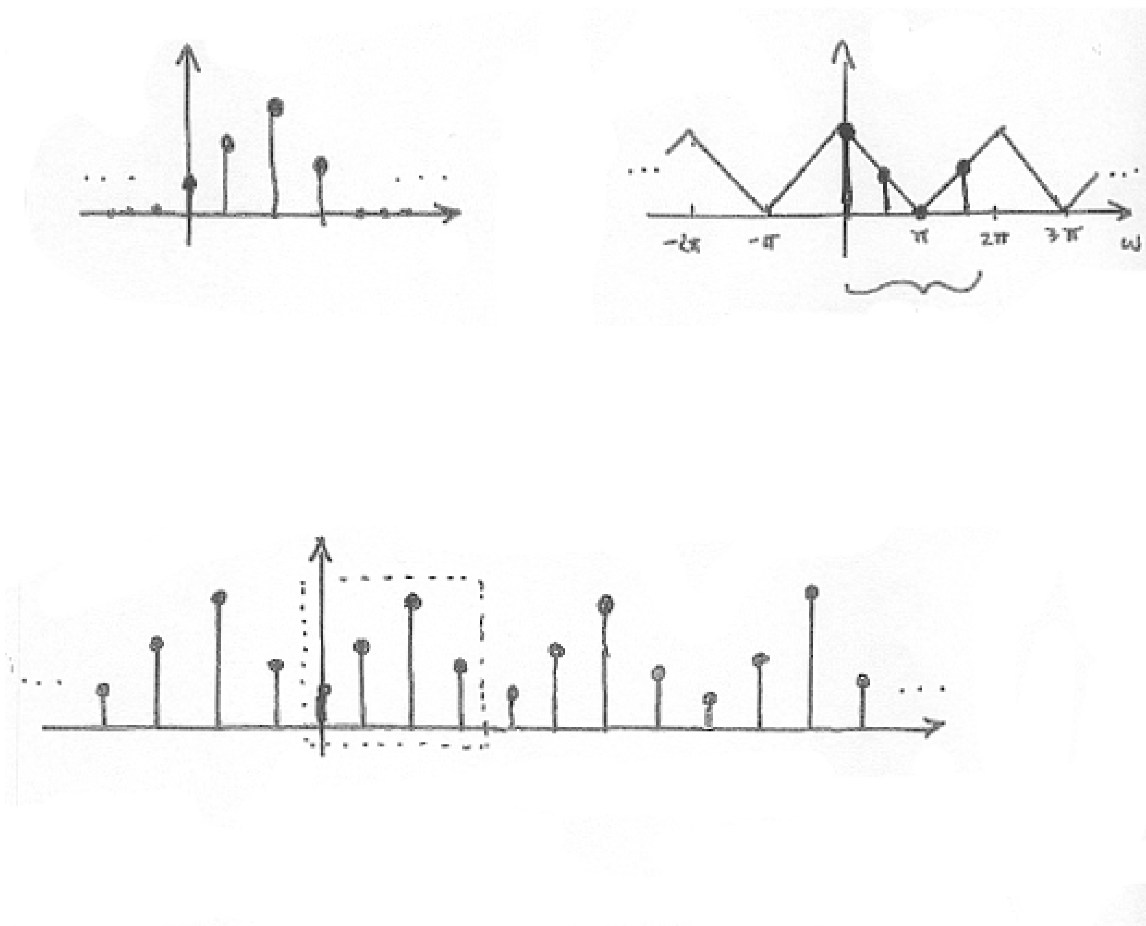

Figure 8

NOTE: When we deal with the DFT, we need to remember that, in effect, this treats the signal as an N-periodic sequence.

4 A Sampling Perspective

Think of sampling the continuous function $X(\omega)$, as depicted in Figure 9. $S(\omega)$ will represent the sampling function applied to $X(\omega)$ and is illustrated in Figure 9 as well. This will result in our discrete-time sequence, $X[k]$.

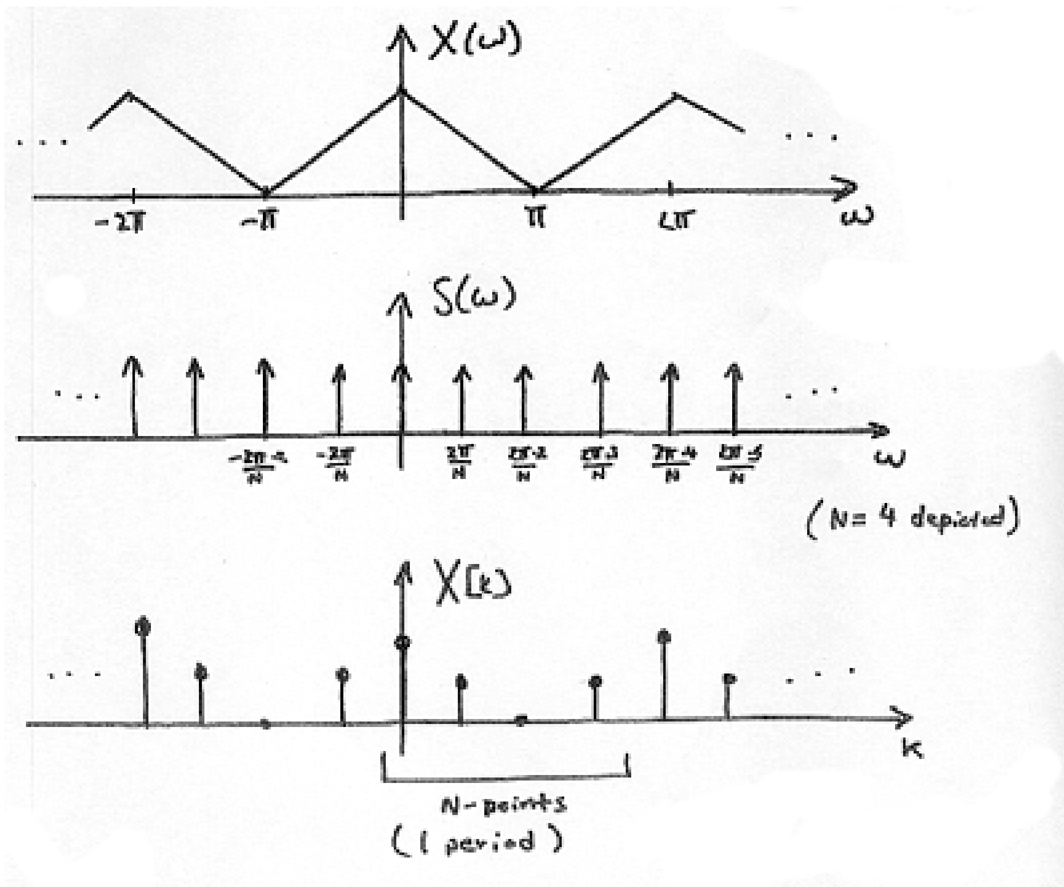

Figure 9

NOTE: Remember the multiplication in the frequency domain is equal to convolution in the time domain!

4.1 Inverse DTFT of S(ω)

$$\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right) \tag{13}$$

Given the above equation, we can take the DTFT and get the following equation:

$$N \sum_{m=-\infty}^{\infty} \delta\left[n - mN\right] \equiv S\left[n\right] \tag{14}$$

Exercise 2 *(Solution on p. 11.)*

Why does (14) equal $S\left[n\right]$?

So, in the time-domain we have (Figure 10):

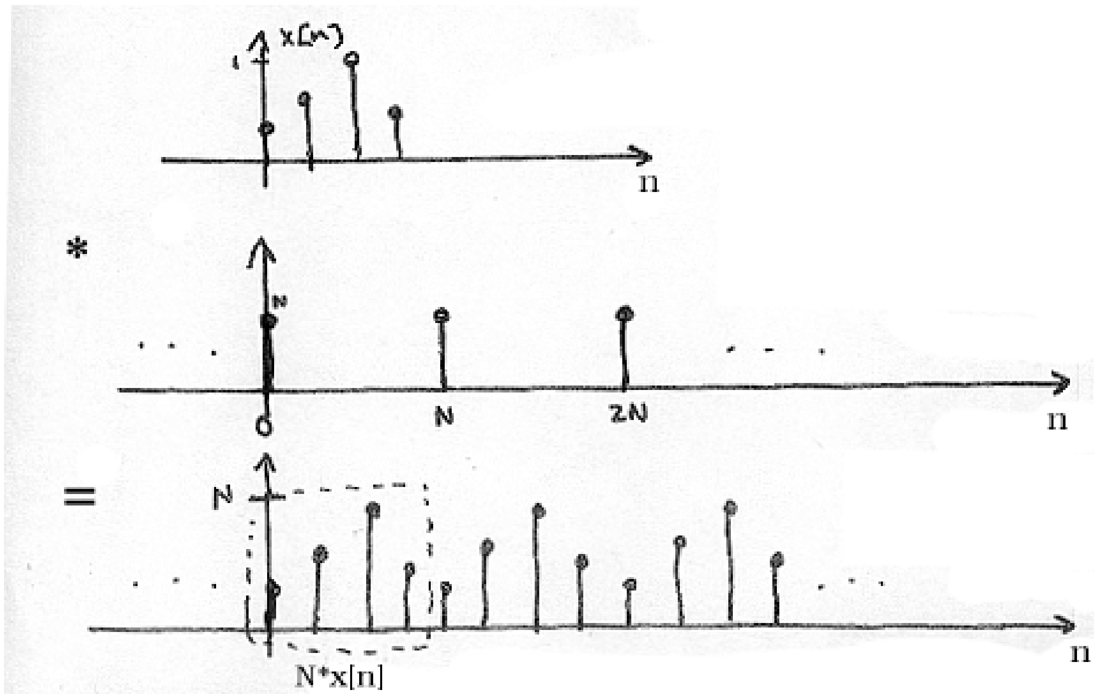

Figure 10

5 Connections

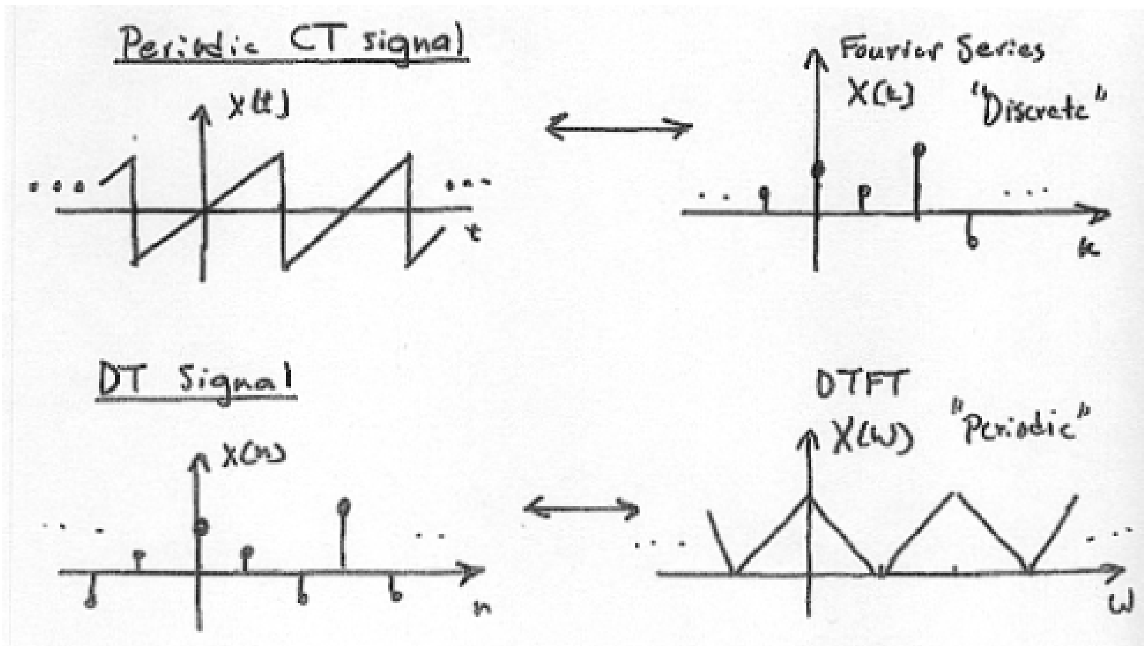

Figure 11

Combine signals in Figure 11 to get signals in Figure 12.

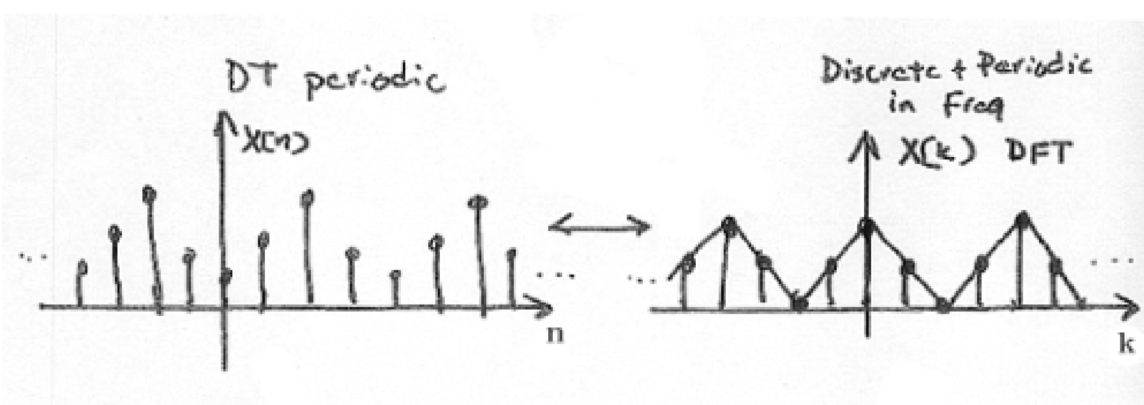

Figure 12

Solutions to Exercises in this Module

Solution to Exercise (p. 4)

$$x[n+N] = ???$$

Solution to Exercise (p. 8)

$S[n]$ is N-periodic, so it has the following Fourier Series[3]:

$$\begin{aligned} c_k &= \frac{1}{N}\int_{-\frac{N}{2}}^{\frac{N}{2}} \delta[n]\, e^{(-i)2\pi\frac{k}{N}n}dn \\ &= \frac{1}{N} \end{aligned} \quad (15)$$

$$S[n] = \sum_{k=-\infty}^{\infty} e^{(-i)2\pi\frac{k}{N}n} \quad (16)$$

where the DTFT of the exponential in the above equation is equal to $\delta\left(\omega - \frac{2\pi k}{N}\right)$.

[3]"Fourier Series: Eigenfunction Approach" <http://cnx.org/content/m10496/latest/>