Approximation Formulae for the Gaussian Error Integral, Q(x)*

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Abstract

This module introduces approximation formulae for the Gaussian error Integral

A Gaussian pdf with unit variance is given by:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{1}$$

The probability that a signal with a pdf given by $f(x)$ lies above a given threshold x is given by the Gaussian Error Integral or Q function:

$$Q(x) = \int_x^{\infty} f(u)\, du \tag{2}$$

There is no analytical solution to this integral, but it has a simple relationship to the error function, $\text{erf}(x)$, or its complement, $\text{erfc}(x)$, which are tabulated in many books of mathematical tables.

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{3}$$

and

$$\begin{aligned} \text{erfc}(x) &= 1 - \text{erf}(x) \\ &= \tfrac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du \end{aligned} \tag{4}$$

Therefore,

$$\begin{aligned} Q(x) &= \tfrac{1}{2} \text{erfc}\left(\tfrac{x}{\sqrt{2}}\right) \\ &= \tfrac{1}{2} \left(1 - \text{erf}\left(\tfrac{x}{\sqrt{2}}\right)\right) \end{aligned} \tag{5}$$

Note that $\text{erf}(0) = 0$ and $\text{erf}(\infty) = 1$, and therefore $Q(0) = 0.5$ and $Q(x) \to 0$ very rapidly as x becomes large.

It is useful to derive simple approximations to $Q(x)$ which can be used on a calculator and avoid the need for tables.

*Version 2.4: Jun 7, 2005 4:20 pm +0000

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Let $v = u - x$, then:

$$\begin{array}{rcl} Q(x) & = & \int_0^\infty f(v+x)\,dv \\ & = & \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{v^2+2vx+x^2}{2}}\,dv \\ & = & \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{-(vx)}e^{-\frac{v^2}{2}}\,dv \end{array} \quad (6)$$

Now if $x \gg 1$, we may obtain an approximate solution by replacing the $e^{-\frac{v^2}{2}}$ term in the integral by unity, since it will initially decay much slower than the $e^{-(vx)}$ term. Therefore

$$Q(x) < \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{-(vx)}dv = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x} \quad (7)$$

This approximation is an upper bound, and its ratio to the true value of $Q(x)$ becomes less than 1.1 only when $x > 3$, as shown in Figure 1. We may obtain a much better approximation to $Q(x)$ by altering the denominator above from ($\sqrt{2\pi}x$) to ($1.64x + \sqrt{0.76x^2+4}$) to give:

$$Q(x) \simeq \frac{e^{-\frac{x^2}{2}}}{1.64x + \sqrt{0.76x^2+4}} \quad (8)$$

This improved approximation gives a curve indistinguishable from $Q(x)$ in Figure 1 and its ratio to the true $Q(x)$ is now within $\pm(0.3\%)$ of unity for all $x \geq 0$ as shown in Figure 2. This accuracy is sufficient for nearly all practical problems.

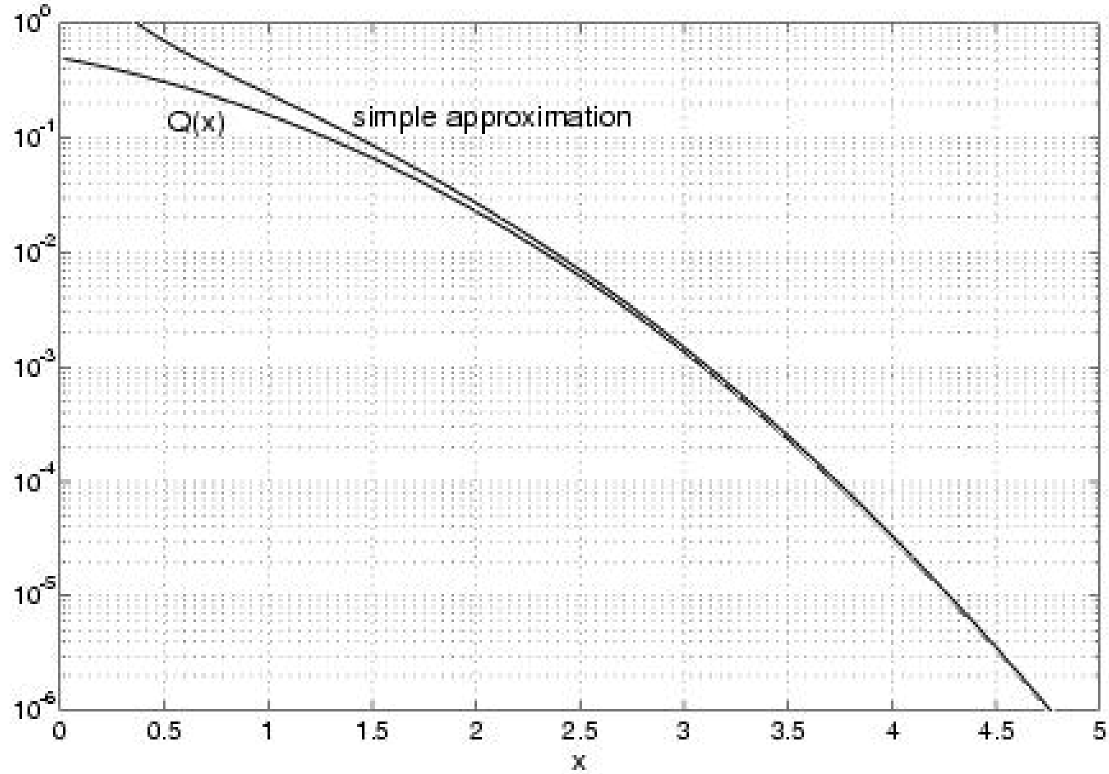

Figure 1: $Q(x)$ and the simple approximation of (7).

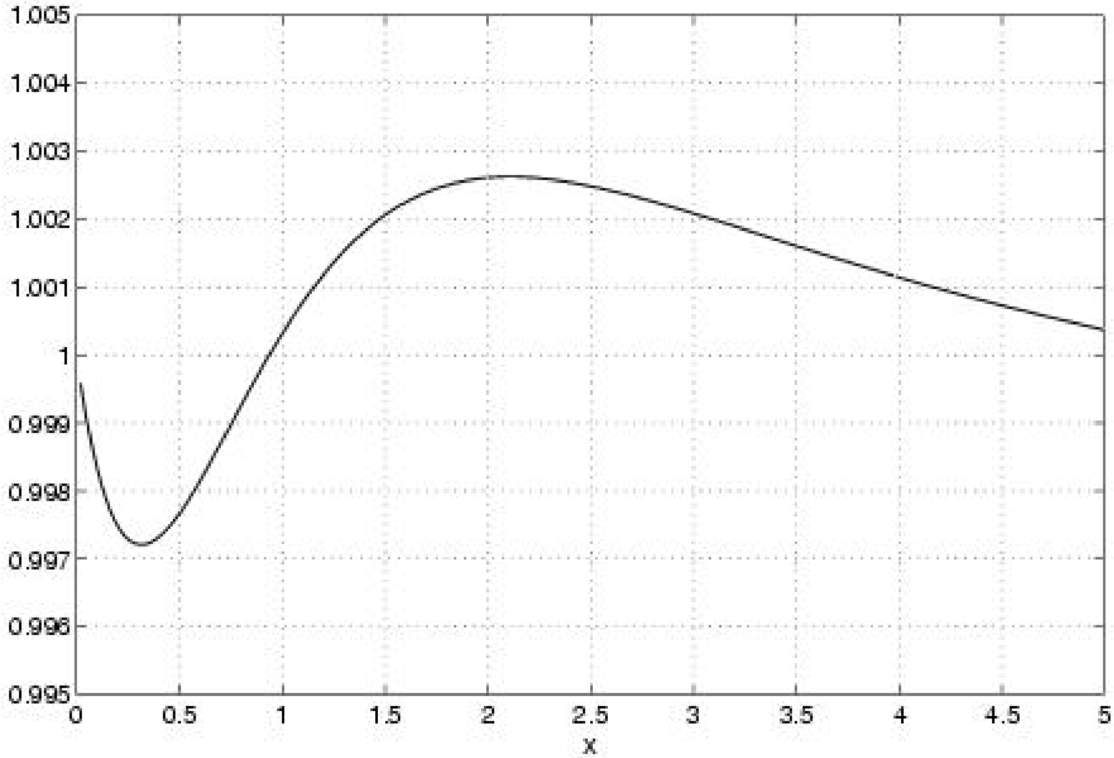

Figure 2: The ration of the improved approximation of $Q(x)$ in (8) to the true value, obtained by numerical integration.