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# Wavelets, Splines, and the Reproduction of Polynomials\*

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### Abstract

On well-known property of valid scaling functions in wavelet analysis is that the scaling functions can perfectly reproduce a polynomial of degree no more than the wavelet smoothness. This property can be easily derived using the Unser-Blu Scaling Function / Spline Factorization Theorem.

Recall from spline theory that fractional B-splines reproduce polynomials of order less than or equal to the ceiling of the spline order. This means that any polynomial of a certain order can be expressed as a linear combination of B-splines; that is, the B-splines form a basis for polynomials. Specifically, for some p which is a nonnegative integer no greater than  $\alpha + 1$ , we have

$$\sum_{kk\in\mathbb{Z}} k^p \beta_+^{\alpha}(x-k) = x^p + a_{p,1} x^{p-1} + \dots + a_{p,p-1} x + a_{p,p}$$
(1)

The collection of all these polynomials for all p no greater than  $\alpha + 1$  forms a basis for polynomials of order no greater than  $\alpha + 1$ .

This combined with the Unser-Blu Scaling Function/Spline Factorization Theorem leads to a straighforward proof of the fact that scaling functions reproduce polynomials up to a degree proportional to their smoothness.

## **Proposition 1:**

Let  $\phi_0$  be any distribution <sup>1</sup> such that  $\int \phi_0(x) dx = 1$  adn  $\int x^i \phi_0(x) dx < \infty$ , for all  $i \in \{1, ..., n\}$ , where  $n = \lceil \alpha \rceil$ . Then  $\phi(x) = \beta_+^{\alpha} * \phi_0(x)$  reproduces polynomials of degree lesser or equal to  $n = \lceil \alpha \rceil$ . **Proof:** 

1. First note that

$$\sum_{kk\in\mathbb{Z}} k^p \phi(x-k) = \sum_{kk\in\mathbb{Z}} k^p \beta_+^{\gamma-1} * \phi_0(x-k)$$
 (2)

$$\sum_{kk\in\mathbb{Z}} k^p \phi(x-k) = \phi_0 * \sum_{k=0}^p a_{p,k} x^{p-k}$$
(3)

2. 
$$\phi_0(x) * x^p = x^p + b_1 x^{p-1} + \dots + b_p$$

<sup>\*</sup>Version 2.3: Jun 7, 2005 3:43 pm -0500

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 $<sup>^{1}</sup>$  http://mathworld.wolfram.com/GeneralizedFunction.html