K-Regularity of Scaling Filter[*]

Feng Qiao
Rachael Milam

Definition 1: K-Regularity
A unitary 2-band scaling filter is said to be K-regular if it has a polynomial factor of the form $P^K(z)$, with $P(z) = \frac{1+z^{-1}}{2}$.

$$H_0(z) = \left(\frac{1+z^{-1}}{2}\right)^K Q(z) \tag{1}$$

Here a unitary filter is defined as a FIR filter with coefficients $h[n]$ from the basic recursive equation $\phi(t) = \sum_{nn} h[n] \sqrt{2}\phi(2t-n)$ satisfying the admissibility conditions and orthogonality conditions, i.e.,

$$\sum_{nn} h[n] = \sqrt{2}$$

and

$$\sum_{kk} h[k] h[k+2m] = \delta[m]$$

NOTE: Any unitary filter is at least 1-regular according to the above definition.

NOTE: In what range can K take its value?

NOTE: The admissibility conditions and orthogonality conditions take $\frac{N}{2}+1$ degrees of freedom out of the total N degrees of freedom in designing the filter $h[n]$, where N is the length of the filter.

Since K is at least 1, K-regularity is equivalent to a set of $K-1$ linear constraints on the scaling filter, this takes another $K-1$ degrees of freedom. So we have $1 \le K \le \frac{N}{2}$.

K-regularity describes the flatness of $H(\omega)$, the Fourier transform of $h[n]$, at $\omega = 0$ and $\omega = \pi$. The flatness at these two locations indicates how well the scaling filter performs as a low-pass filter, and is related to the smoothness of corresponding scaling and wavelet functions. Figure 1 shows the frequency response of Daubechies scaling filters with increasing regularities.
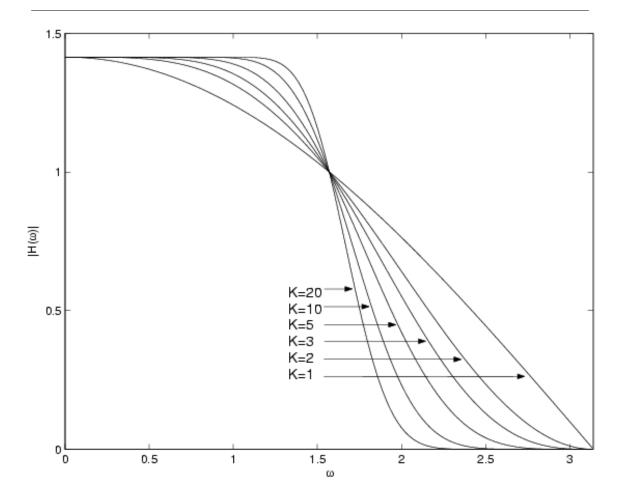

Figure 1: Frequency response of Daubechies scaling filters

It can be proved that putting K-regularity on scaling filter is equivalent to the following statements:

1. All moments of the wavelet filters are zero, $\mu_1[k] = 0$, for $k \in \{0, 1, \ldots, K-1\}$.
2. All moments of the wavelets are zero, $m_1[k] = 0$, for $k \in \{0, 1, \ldots, K-1\}$.
3. The partial moments of the scaling filter are equal for $k \in \{0, 1, \ldots, K-1\}$.
4. The frequency response of the scaling filter has a zero of order K at $\omega = \pi$.
5. The kth derivative of the magnitude-squared frequency response of the scaling filter is zero at $\omega = 0$, for $k \in \{1, 2, \ldots, 2K-1\}$.
6. All polynomial sequences up to degree $K-1$ can be expressed as a linear combination of shifted scaling filters.
7. All polynomials of degree up to $K-1$ can be expressed as a linear combination of shifted scaling functions at any scale.

Statement 1-3 relate the regularity of scaling filter to vanishing wavelet moments and the moments of scaling filter. Statement 4 and 5 describes the flatness of the frequency response of the scaling filter. The last two statements show the power of polynomial representation as a consequence of K-regularity.

Daubechies filters utilize all the remaining $\frac{N}{2}-1$ degrees of freedom to get maximum number of vanishing wavelet moments, i.e., $K = \frac{N}{2}$ for Daubechies filters.

Here[1] is a LabVIEW VI showing the Daubechies scaling and wavelet functions, along with the frequency response of scaling and wavelet filters for $k \in \{2, 3, \ldots, 14\}$.

---

[1] http://cnx.org/content/m11158/latest/daub.vi