Random Vectors*

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A **random vector**X is an ordered sequence of random variables $X = (X_1, \ldots, X_L)^T$. The density function of a random vector is defined in a manner similar to that for pairs of random variables considered previously. The expected value of a random vector is the vector of expected values.

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x p_X(x)\, dx \\ &= \begin{pmatrix} E[X_1] \\ \vdots \\ E[X_L] \end{pmatrix} \end{aligned} \tag{1}$$

The **covariance matrix**K_X is an $L \times L$ matrix consisting of all possible covariances among the random vector's components.

$$\forall i, j, i \wedge j \in \{1, \ldots, L\} : \left({K^X}_{i,j} = \operatorname{cov}(X_i, X_j) = E\left[X_i \overline{X_j}\right] - E[X_i] E\left[\overline{X_j}\right] \right) \tag{2}$$

Using matrix notation, the covariance matrix can be written as $K_X = E\left[(X - E[X])(X - E[X])^T\right]$. Using this expression, the covariance matrix is seen to be a symmetric matrix and, when the random vector has no zero-variance component, its covariance matrix is positive-definite. Note in particular that when the random variables are real-valued, the diagonal elements of a covariance matrix equal the variances of the components: ${K^X}_{i,i} = \sigma^2_{X_i}$. **Circular random vectors** are complex-valued with uncorrelated, identically distributed, real and imaginary parts. In this case, $E\left[(|X_i|)^2\right] = 2\sigma^2_{X_i}$, and $E\left[{X_i}^2\right] = 0$. By convention, $\sigma^2_{X_i}$ denotes the variance of the real (or imaginary) parts. The characteristic function of a real-valued random vector is defined to be

$$\Phi_{\mathbf{X}}(i\nu) = E\left[e^{i\nu^T X}\right] \tag{3}$$

The maximum of a random vector is a random variables whose probability density is usually quite different from the distributions of the vector's components. The probability that the maximum is less than some number μ is equal to the probability that **all** of the components are less than μ.

$$Pr[max\{X\} < \mu] = P_X(\mu, \ldots, \mu) \tag{4}$$

Assuming that the components of X are statistically independent, this expression becomes

$$Pr[max\{X\} < \mu] = \prod_{i=1}^{\dim(X)} P_{\mathrm{Xi}}(\mu) \tag{5}$$

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