Karhunen-Loeve Expansion*

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The representation of the process, $X(t)$, is the sequence of random variables X_i. The choice basis of $\{\phi_i(t)\}$ is unrestricted. Of particular interest is to restrict the basis functions to those which make the $\{X_i\}$**uncorrelated** random variables. When this requirement is satisfied, the resulting representation of $X(t)$ is termed the **Karhunen-Loève** expansion. Mathematically, we require $\forall i,j,i \neq j : (E[X_i X_j] = E[X_i]E[X_j])$. This requirement can be expressed in terms of the correlation function of $X(t)$.

$$E[X_i X_j] = E\left[\int_0^T X(\alpha)\phi_i(\alpha)\,d\alpha \int_0^T X(\beta)\phi_j(\beta)\,d\beta\right] = \int_0^T \int_0^T \phi_i(\alpha)\phi_j(\beta) R_X(\alpha,\beta)\,d\alpha d\beta$$

As $E[X_i]$ is given by

$$E[X_i] = \int_0^T m_X(\alpha)\phi_i(\alpha)\,d\alpha$$

our requirement becomes

$$\forall i,j,i \neq j : \left(\int_0^T \int_0^T \phi_i(\alpha)\phi_j(\beta) R_X(\alpha,\beta)\,d\alpha d\beta = \int_0^T m_X(\alpha)\phi_i(\alpha)\,d\alpha \int_0^T m_X(\beta)\phi_j(\beta)\,d\beta\right) \tag{1}$$

Simple manipulations result in the expression

$$\forall i,j,i \neq j : \left(\int_0^T \phi_i(\alpha) \int_0^T K_X(\alpha,\beta)\phi_j(\beta)\,d\beta d\alpha = 0\right) \tag{2}$$

When $i = j$, the quantity $E\left[{X_i}^2\right] - (E[X_i])^2$ is just the variance of X_i. Our requirement is obtained by satisfying

$$\int_0^T \phi_i(\alpha) \int_0^T K_X(\alpha,\beta)\phi_j(\beta)\,d\beta d\alpha = \lambda_i \delta_{ij}$$

or

$$\forall i,j,i \neq j : \left(\int_0^T \phi_i(\alpha) g_j(\alpha)\,d\alpha = 0\right) \tag{3}$$

where

$$g_j(\alpha) = \int_0^T K_X(\alpha,\beta)\phi_j(\beta)\,d\beta$$

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Furthermore, this requirement must hold for each j which differs from the choice of i. A choice of a function $g_j(\alpha)$ satisfying this requirement is a function which is proportional to $\phi_j(\alpha)$: $g_j(\alpha) = \lambda_j \phi_j(\alpha)$. Therefore,

$$\int_0^T K_X(\alpha, \beta)\, \phi_j(\beta)\, d\beta = \lambda_j \phi_j(\alpha) \tag{4}$$

The $\{\phi_i\}$ which allow the representation of $X(t)$ to be a sequence of uncorrelated random variables must satisfy this integral equation. This type of equation occurs often in applied mathematics; it is termed the **eigenequation**. The sequences $\{\phi_i\}$ and $\{\lambda_i\}$ are the eigenfunctions and eigenvalues of $K_X(\alpha, \beta)$, the covariance function of $X(t)$. It is easily verified that:

$$K_X(t, u) = \sum_{i=1}^{\infty} \lambda_i \phi_i(t)\, \phi_i(u)$$

This result is termed **Mercer's Theorem**.

The approach to solving for the eigenfunction and eigenvalues of $K_X(t, u)$ is to convert the integral equation into an ordinary differential equation which can be solved. This approach is best illustrated by an example.

Example 1

$K_X(t, u) = \sigma^2 min\{t, u\}$. The eigenequation can be written in this case as

$$\sigma^2 \left(\int_0^t u\phi(u)\, du + t \int_t^T \phi(u)\, du \right) = \lambda \phi(t)$$

Evaluating the first derivative of this expression,

$$\sigma^2 t\phi(t) + \sigma^2 \int_t^T \phi(u)\, du - \sigma^2 t\phi(t) = \lambda \frac{d\phi(t)}{dt}$$

or

$$\sigma^2 \int_t^T \phi(u)\, du = \lambda \frac{d\phi}{dt}$$

Evaluating the derivative of the last expression yields the simple equation

$$-\left(\sigma^2 \phi(t)\right) = \lambda \frac{d^2\phi}{dt^2}$$

This equation has a general solution of the form $\phi(t) = A\sin\left(\frac{\sigma}{\sqrt{\lambda}}t\right) + B\cos\left(\frac{\sigma}{\sqrt{\lambda}}t\right)$. It is easily seen that B must be zero. The amplitude A is found by requiring $\| \phi \| = 1$. To find λ, one must return to the original integral equation. Substituting, we have

$$\sigma^2 A \int_0^t u\sin\left(\frac{\sigma}{\sqrt{\lambda}}u\right) du + \sigma^2 tA \int_t^T \sin\left(\frac{\sigma}{\sqrt{\lambda}}u\right) du = \lambda A\sin\left(\frac{\sigma}{\sqrt{\lambda}}t\right)$$

After some manipulation, we find that

$$\forall t, t \in [0, T) : \left(A\lambda\sin\left(\frac{\sigma}{\sqrt{\lambda}}t\right) - A\sigma t\sqrt{\lambda}\cos\left(\frac{\sigma}{\sqrt{\lambda}}T\right) = \lambda A\sin\left(\frac{\sigma}{\sqrt{\lambda}}t\right)\right)$$

or

$$\forall t, t \in [0, T) : \left(A\sigma t\sqrt{\lambda}\cos\left(\frac{\sigma}{\sqrt{\lambda}}T\right) = 0\right)$$

Therefore, $\forall n, n = \{1, 2, \dots\} : \left(\frac{\sigma}{\sqrt{\lambda}} T = (n - 1/2)\pi\right)$ and we have

$$\lambda_n = \frac{\sigma^2 T^2}{(n + 1/2)^2 \pi^2}$$

$$\phi_n(t) = \left(\frac{2}{T}\right)^{1/2} \sin\left(\frac{(n + 1/2)\pi t}{T}\right)$$

The Karhunen-Loève expansion has several important properties.

- The eigenfunctions of a positive-definite covariance function constitute a complete set. One can easily show that these eigenfunctions are also mutually orthogonal with respect to both the usual inner product and with respect to the inner product derived from the covariance function.
- If $X(t)$ Gaussian, X_i are Gaussian random variables. As the random variables $\{X_i\}$ are uncorrelated and Gaussian, the $\{X_i\}$ comprise a sequence of statistically independent random variables.
- Assume $K_X(t, u) = \frac{N_0}{2}\delta(t - u)$: the stochastic process $X(t)$ is white. Then

  $$\int \frac{N_0}{2}\delta(t - u)\phi(u)\,du = \lambda\phi(t)$$

  for all $\phi(t)$. Consequently, if $\lambda_i = \frac{N_0}{2}$, this constraint equation is satisfied **no matter what choice is made for the orthonormal set**$\{\phi_i(t)\}$. Therefore, the representation of white, Gaussian processes consists of a sequence of statistically independent, identically-distributed (mean zero and variance $\frac{N_0}{2}$) Gaussian random variables. This example constitutes the simplest case of the Karhunen-Loève expansion.