Transmission Line Equation*

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Abstract

This document continues derivation of Transmission Line equations. It also introduces characteristic impedance.

We need to solve the **telegrapher's equations**,

$$\frac{\partial V(x,t)}{\partial x} = -\left(\mathbf{L}\frac{\partial V(x,t)}{\partial t}\right) \tag{1}$$

$$\frac{\partial I(x,t)}{\partial x} = -\left(\mathbf{C}\frac{\partial V(x,t)}{\partial t}\right) \tag{2}$$

The way we will proceed to a solution, and the way you always proceed when confronted with a pair of equations such as these, is to take a spatial derivative of one equation, and then substitute the second equation in for the spatial derivative in the first and you end up with...well, let's try it and see.

Taking a derivative with respect to x of (1)

$$\frac{\partial^2 V(x,t)}{\partial x^2} = -\left(\mathbf{L}\frac{\partial^2 I(x,t)}{\partial t \partial x}\right) \tag{3}$$

Now we substitute in for $\frac{\partial I(x,t)}{\partial x}$ from (2)

$$\frac{\partial^2 V(x,t)}{\partial x^2} = \mathbf{LC}\frac{\partial^2 V(x,t)}{\partial t^2} \tag{4}$$

It should be **very** easy for you to derive

$$\frac{\partial^2 I(x,t)}{\partial x^2} = \mathbf{LC}\frac{\partial^2 I(x,t)}{\partial t^2} \tag{5}$$

Oh, I know you all **love** differential equations! Well, let's take a look at these and just **think** for a minute. For **either**$V(x,t)$ or $I(x,t)$, we need to find a function that has some rather stringent requirements. First of all, the function must be of the form such that no matter whether we take its second derivative in space (x) or in time (t), it must end up differing in the way it behaves in x or t by no more than just a constant (**LC**).

In fact, we can be more specific than that. First $V(x,t)$ must have the same functional form for **both** its x and t variation. At most, the two derivatives must differ only by a constant. Let's try a "lucky" guess and let:

$$V(x,t) = V_0 f(x - vt) \tag{6}$$

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where V_0 is the amplitude of the voltage, and f is some function, of a form yet undetermined. Well

$$\frac{\partial f(x-vt)}{\partial t} = -\left(v\frac{df(x-vt)}{d}\right) \tag{7}$$

and

$$\frac{\partial(-(vf(x-vt)))}{\partial t} = v^2\frac{d^2 f(x-vt)}{dt^2} \tag{8}$$

Note also, that

$$\frac{\partial^2 f(x-vt)}{\partial x^2} = \frac{d^2}{dt^2}f(x-vt) \tag{9}$$

Thus, we have

$$\begin{aligned} \frac{\partial^2 V(x,t)}{\partial x^2} &= V_0\frac{d^2 f(x-vt)}{dt^2} \\ &= \frac{1}{\mathbf{LC}}\frac{\partial^2 V(x,t)}{\partial t^2} \\ &= \frac{v^2}{\mathbf{LC}}V_0\frac{d^2 f(x-vt)}{dt^2} \end{aligned} \tag{10}$$

which works as a solution as long as

$$v = \frac{1}{\sqrt{\mathbf{LC}}} \tag{11}$$

So, what is this $f(x-vt)$? We don't know yet what its actual functional form will be, but suppose at some point in time, t_1, the function looks like Figure 1 (f(x) At Some Point In Time).

f(x) At Some Point In Time

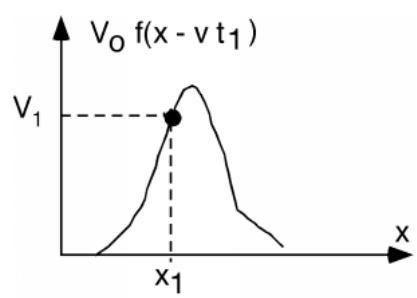

Figure 1: $f(x)$ at time t_1.

At point x_1, the function takes on the value V_1. Now, let's advance to time t_2. We look at the function and we see Figure 2 (f(x) At a Later Point In Time).

f(x) At a Later Point In Time

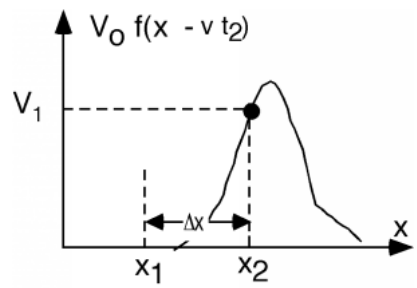

Figure 2: $f(x)$ at a later t_2.

If t increases from t_1 to t_2 then x will have to increase from x_1 to x_2 in order for the argument of f to have the same value, V_1. Thus we find

$$x_1 - vt_1 = x_2 - vt_2 \tag{12}$$

which can be re-written as

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta(x)}{\Delta(t)} \equiv v_p = \frac{1}{\sqrt{\mathbf{LC}}} \tag{13}$$

where $v_p$is the velocity with which the function is moving along the x-axis! (We use the subscript "p" to indicated that what we have here is what is called the **phase velocity**. We will encounter another velocity called the **group velocity** a little later in the course.)

If we had "guessed" an $f(x + vt)$ for our function, it should be pretty easy to see that this would have given us a signal moving in the **minus**x direction, instead of the plus x direction. Thus we shall denote

$$V_{\text{plus}} = V^+ f\left(x - \frac{1}{\sqrt{\mathbf{LC}}}t\right) \tag{14}$$

the **positive** going voltage function and

$$V_{\text{minus}} = V^- f\left(x + \frac{1}{\sqrt{\mathbf{LC}}}t\right) \tag{15}$$

which is the negative going voltage function. Notice that since we are taking the **second** derivative of f with respect to t, we are free to choose either a $\frac{1}{\sqrt{\mathbf{LC}}}$ or a $-\frac{1}{\sqrt{\mathbf{LC}}}$ in front of the time argument inside f. Also note that these are our **only** choices for a solution. As we know from Differential Equations, a second order equation has, at most, two independent solutions.

Since $I(x,t)$ has the **same** differential equation describing its behavior, the solutions for I must also be of the exact same form. Thus we can let

$$I_{\text{plus}} = I^+ f\left(x - \frac{1}{\sqrt{\mathbf{LC}}}t\right) \tag{16}$$

represent the current function which goes in the positive x direction, and

$$I_{\text{minus}} = I^- f\left(x + \frac{1}{\sqrt{\mathbf{LC}}}t\right) \tag{17}$$

represent the negative going current function.

Now, let's take (16) and (14) and substitute them into (1):

$$\frac{V^+}{\sqrt{\mathbf{LC}}} f\left(x - \frac{1}{\sqrt{\mathbf{LC}}} t\right) = \mathbf{L} I^+ f\left(x - \frac{1}{\sqrt{\mathbf{LC}}} t\right) \tag{18}$$

This can be solved for V^+ in terms of I^+.

$$V^+ = \sqrt{\frac{\mathbf{L}}{\mathbf{C}}} I^+ \equiv Z_0 I^+ \tag{19}$$

where $Z_0 = \sqrt{\frac{\mathbf{L}}{\mathbf{C}}}$ is called the **characteristic impedance** of the transmission line. We will leave it as an exercise to the reader to ensure that indeed $\sqrt{\frac{\mathbf{L}}{\mathbf{C}}}$ has units of Ohms. For practice, and understanding about just how these equations work, the reader should ensure him/her self that

$$V^- = -\left(\sqrt{\frac{\mathbf{L}}{\mathbf{C}}} I^-\right) \equiv -\left(Z_0 I^-\right) \tag{20}$$

Note the "subtle" difference here, with a "-" sign in front of the RHS of the equation!

We've been through lots of equations recently, so it is probably worth our while to summarize what we know so far.

1. The telegrapher's equations allow two solutions for the voltage and current on a transmission line. One moves in the x direction and the other moves in the $-x$ direction.
2. Both signals move at a constant velocity v_p given by (21).
3. The voltage and current signals are related to one another by the characteristic impedance Z_0, with (22)

$$v_p = \frac{1}{\sqrt{\mathbf{LC}}} \tag{21}$$

$$Z_0 = \sqrt{\frac{\mathbf{L}}{\mathbf{C}}} \tag{22}$$

$$\frac{V^+}{I^+} = Z_0$$

$$\frac{V^-}{I^-} = -Z_0$$