Proof*

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Abstract

Proof of Shannon's sampling theorem

NOTE: In order to recover the signal $x(t)$ from it's samples exactly, it is necessary to sample $x(t)$ at a rate greater than twice it's highest frequency component.

1 Introduction

As mentioned earlier[1], sampling is the necessary fundament when we want to apply digital signal processing on analog signals.

Here we present the proof of the sampling theorem. The proof is divided in two. First we find an expression for the spectrum of the signal resulting from sampling the original signal $x(t)$. Next we show that the signal $x(t)$ can be recovered from the samples. Often it is easier using the frequency domain when carrying out a proof, and this is also the case here.

Key points in the proof

- We find an equation (8) for the spectrum of the sampled signal
- We find a simple method to reconstruct (14) the original signal
- The sampled signal has a periodic spectrum...
- ...and the period is $2 \times \pi F_s$

2 Proof part 1 - Spectral considerations

By sampling $x(t)$ every T_s second we obtain $x_s(n)$. The inverse fourier transform of this time discrete signal[2] is

$$x_s(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_s\left(e^{i\omega}\right) e^{i\omega n} d\omega \tag{1}$$

For convenience we express the equation in terms of the real angular frequency Ω using $\omega = \Omega T_s$. We then obtain

$$x_s(n) = \frac{T_s}{2\pi} \int_{\frac{-\pi}{T_s}}^{\frac{\pi}{T_s}} X_s\left(e^{i\Omega T_s}\right) e^{i\Omega T_s n} d\Omega \tag{2}$$

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[1] "Introduction" <http://cnx.org/content/m11419/latest/#s1p1>

[2] "Discrete time signals" <http://cnx.org/content/m11476/latest/>

The inverse fourier transform of a continuous signal is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(i\Omega) e^{i\Omega t} d\Omega \tag{3}$$

From this equation we find an expression for $x(nT_s)$

$$x(nT_s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(i\Omega) e^{i\Omega n T_s} d\Omega \tag{4}$$

To account for the difference in region of integration we split the integration in (4) into subintervals of length $\frac{2\pi}{T_s}$ and then take the sum over the resulting integrals to obtain the complete area.

$$x(nT_s) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{\frac{(2k-1)\pi}{T_s}}^{\frac{(2k+1)\pi}{T_s}} X(i\Omega) e^{i\Omega n T_s} d\Omega \tag{5}$$

Then we change the integration variable, setting $\Omega = \eta + \frac{2\times\pi k}{T_s}$

$$x(nT_s) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{\frac{-\pi}{T_s}}^{\frac{\pi}{T_s}} X\left(i\left(\eta + \frac{2\times\pi k}{T_s}\right)\right) e^{i\left(\eta + \frac{2\times\pi k}{T_s}\right) n T_s} d\eta \tag{6}$$

We obtain the final form by observing that $e^{i2\times\pi k n} = 1$, reinserting $\eta = \Omega$ and multiplying by $\frac{T_s}{T_s}$

$$x(nT_s) = \frac{T_s}{2\pi} \int_{\frac{-\pi}{T_s}}^{\frac{\pi}{T_s}} \sum_{k=-\infty}^{\infty} \frac{1}{T_s} X\left(i\left(\Omega + \frac{2\times\pi k}{T_s}\right)\right) e^{i\Omega n T_s} d\Omega \tag{7}$$

To make $x_s(n) = x(nT_s)$ for all values of n, the integrands in (2) and (7) have to agreee, that is

$$X_s\left(e^{i\Omega T_s}\right) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X\left(i\left(\Omega + \frac{2\pi k}{T_s}\right)\right) \tag{8}$$

This is a central result. We see that the digital spectrum consists of a sum of shifted versions of the original, analog spectrum. Observe the periodicity!

We can also express this relation in terms of the digital angular frequency $\omega = \Omega T_s$

$$X_s\left(e^{i\omega}\right) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X\left(i\frac{\omega + 2\times\pi k}{T_s}\right) \tag{9}$$

This concludes the first part of the proof. Now we want to find a reconstruction formula, so that we can recover $x(t)$ from $x_s(n)$.

3 Proof part II - Signal reconstruction

For a bandlimited[3] signal the inverse fourier transform is

$$x(t) = \frac{1}{2\pi} \int_{\frac{-\pi}{T_s}}^{\frac{\pi}{T_s}} X(i\Omega) e^{i\Omega t} d\Omega \tag{10}$$

In the interval we are integrating we have: $X_s\left(e^{i\Omega T_s}\right) = \frac{X(i\Omega)}{T_s}$. Substituting this relation into (10) we get

$$x(t) = \frac{T_s}{2\pi} \int_{\frac{-\pi}{T_s}}^{\frac{\pi}{T_s}} X_s\left(e^{i\Omega T_s}\right) e^{i\Omega t} d\Omega \tag{11}$$

[3] "Illustrations", Figure 2 <http://cnx.org/content/m11443/latest/#f2>

Using the DTFT[4] relation for $X_s\left(e^{i\Omega T_s}\right)$ we have

$$x(t) = \frac{T_s}{2\pi}\int_{\frac{-\pi}{T_s}}^{\frac{\pi}{T_s}} \sum_{n=-\infty}^{\infty} x_s(n)\, e^{-(i\Omega n T_s)} e^{i\Omega t} d\Omega \tag{12}$$

Interchanging integration and summation (under the assumption of convergence) leads to

$$x(t) = \frac{T_s}{2\pi}\sum_{n=-\infty}^{\infty} x_s(n) \int_{\frac{-\pi}{T_s}}^{\frac{\pi}{T_s}} e^{i\Omega(t-nT_s)} d\Omega \tag{13}$$

Finally we perform the integration and arrive at the important reconstruction formula

$$x(t) = \sum_{n=-\infty}^{\infty} x_s(n)\, \frac{\sin\left(\frac{\pi}{T_s}(t-nT_s)\right)}{\frac{\pi}{T_s}(t-nT_s)} \tag{14}$$

(Thanks to R.Loos for pointing out an error in the proof.)

4 Summary

NOTE: $X_s\left(e^{i\Omega T_s}\right) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\left(i\left(\Omega + \frac{2\pi k}{T_s}\right)\right)$

NOTE: $x(t) = \sum_{n=-\infty}^{\infty} x_s(n)\, \frac{\sin\left(\frac{\pi}{T_s}(t-nT_s)\right)}{\frac{\pi}{T_s}(t-nT_s)}$

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Go to Introduction[5]; Illustrations[6]; Matlab Example[7]; Hold operation[8]; Aliasing applet[9]; System view[10]; Exercises[11] ?

[4] "Table of Formulas" <http://cnx.org/content/m11450/latest/>
[5] "Introduction" <http://cnx.org/content/m11419/latest/>
[6] "Illustrations" <http://cnx.org/content/m11443/latest/>
[7] "Sampling and reconstruction with Matlab" <http://cnx.org/content/m11549/latest/>
[8] "Hold operation" <http://cnx.org/content/m11458/latest/>
[9] "Aliasing Applet" <http://cnx.org/content/m11448/latest/>
[10] "Systems view of sampling and reconstruction" <http://cnx.org/content/m11465/latest/>
[11] "Exercises" <http://cnx.org/content/m11442/latest/>