Fundamental Probability Density Functions and Properties*

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1 Gaussian (Normal)

The **Gaussian** probability density function (PDF) (also referred to as the **normal** PDF) for a scalar random variable x is defined as

$$\forall x, x \in (-\infty, \infty) : \left(p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{1}{2\sigma^2}(x-\mu)^2\right)} \right) \tag{1}$$

where μ is the mean and σ^2 is the variance of x. It is denoted by $\mathcal{N}\left(\mu, \sigma^2\right)$ and we say that $x \sim \mathcal{N}\left(\mu, \sigma^2\right)$, where "$\sim$" means "is distributed according to."

The cumulative distribution function (CDF) for $\mu = 0$ and $\sigma^2 = 1$, for which the PDF is termed a **standard normal** PDF, is defined as

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}t^2\right)} dt \tag{2}$$

A more convenient description, which is termed the **right-tail probability** and is the probability of exceeding a given value, is defined as $Q(x) = 1 - \Phi(x)$, where

$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}t^2\right)} dt \tag{3}$$

An approximation that is sometimes useful is

$$Q(x) \simeq \frac{1}{\sqrt{2\pi}x} e^{-\left(\frac{1}{2}x^2\right)} \tag{4}$$

It is shown in Figure 1 (Approximation to Q Function) along with the exact value of $Q(x)$. The approximation is quite accurate for $x > 4$.

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Approximation to Q Function

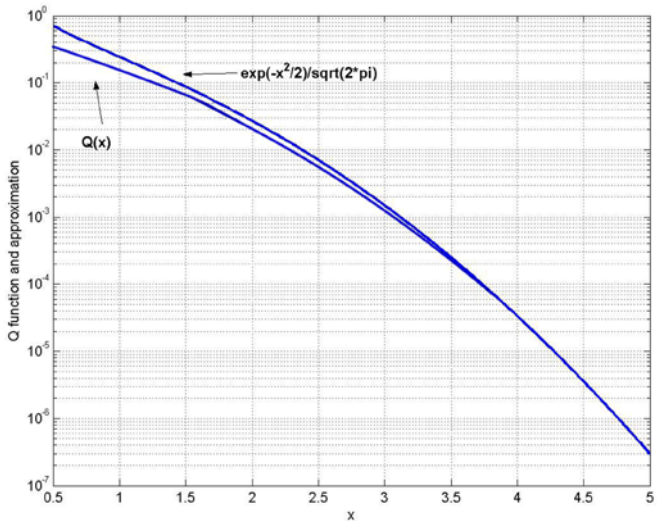

Figure 1

The **multivariate Gaussian** PDF of an $n \times 1$ random vector x is defined as

$$p(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det C}} e^{-\left(\frac{1}{2}(x-\mu)^T C^{-1}(x-\mu)\right)} \tag{5}$$

where μ is the mean vector and C is the covariance matrix and is denoted by $\mathcal{N}(\mu, C)$. It is assumed that C is positive definite and hence C^{-1} exists. The mean vector is defined as

$$\forall i : (\mu_i = E[x_i]) \tag{6}$$

and the covariance matrix as

$$\forall \text{ij}, i \in \{1, 2, \ldots, n\} \ \wedge \ j \in \{1, 2, \ldots, n\} : (C_{i,j} = E[(x_i - E[x_i])(x_j - E[x_j])]) \tag{7}$$

or in more compact form

$$C = E\left[(x - E[x])(x - E[x])^T\right] \tag{8}$$

2 Chi-Squared (Central)

A **chi-squared** PDF with ν degrees of freedom is defined as

$$p(x) = \begin{cases} \frac{1}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\left(\frac{1}{2}x\right)} & \text{if } x > 0 \\ 0 \text{ if } x < 0 \end{cases} \tag{9}$$

and is denoted by ${\chi_\nu}^2$. The degrees of freedom ν is assumed to be an integer with $\nu \geq 1$. The function $\Gamma(u)$ is the Gamma function which is defined as

$$\Gamma(u) = \int_0^\infty t^{u-1} e^{-t} dt \tag{10}$$

The relations $\Gamma(u) = (u-1)\Gamma(u-1)$ for any u, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, and $\Gamma(n) = (n-1)!$ for n an integer.

The chi-squared PDF arises as the PDF of x where $x = \sum_{i=1}^{\nu} {x_i}^2$ if $x_i \sim \mathcal{N}(0,1)$ and the x_i's are independent and identically distributed (IID). By the latter we mean that each x_i is independent of the others and each x_i has the same PDF (identically distributed). The mean and variance are

$$E[x] = \nu \tag{11}$$

$$\sigma(x)^2 = 2\nu \tag{12}$$

A specific case of interest occurs when $\nu = 2$ so that

$$p(x) = \begin{cases} \frac{1}{2}e^{-\left(\frac{1}{2}x\right)} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{13}$$

and is referred to as an **exponential** PDF (Figure 2 (PDF for Chi-Squared Random Variable)).

PDF for Chi-Squared Random Variable

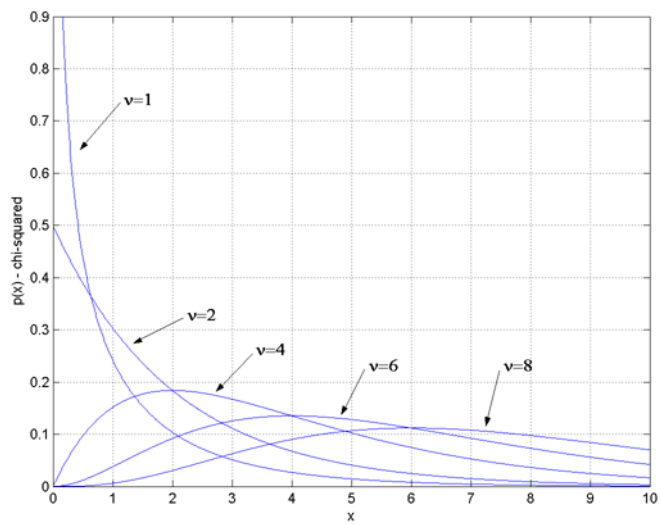

Figure 2

3 F (Central)

The **F** PDF arises as the ratio of two independent χ^2 random variables. Specifically, if

$$x = \frac{\frac{x_1}{\nu_1}}{\frac{x_2}{\nu_2}} \tag{14}$$

where $x_1 \sim {\chi_{\nu_1}}^2$, $x_2 \sim {\chi_{\nu_2}}^2$, and x_1 and x_2 are independent, then x has the **F** PDF. It is given by

$$p\left(x\right) = \begin{cases} \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \frac{x^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1}{\nu_2}x\right)^{\frac{\nu_1+\nu_2}{2}}} & \text{if } x > 0 \\ 0 \text{ if } x < 0 \end{cases} \tag{15}$$

where $B\left(u,v\right)$ is the Beta function, which can be related to the Gamma functions as

$$B\left(u,v\right) = \frac{\Gamma\left(u\right)\Gamma\left(v\right)}{\Gamma\left(u+v\right)} \tag{16}$$

The PDF is denoted by F_{ν_1,ν_2} as an **F** PDF with ν_1 numerator degrees of freedom and ν_2 denominator degrees of freedom.

The mean and variance are

$$\forall \nu 2, \nu_2 > 2 : \left(E\left[x\right] = \frac{\nu_2}{\nu_2 - 2}\right) \tag{17}$$

$$\forall \nu 2, \nu_2 > 4 : \left(\sigma\left(x\right)^2 = \frac{2{\nu_2}^2\left(\nu_1+\nu_2-2\right)}{\nu_1\left(\nu_2-2\right)^2\left(\nu_2-4\right)}\right) \tag{18}$$