The Q-function*

Clayton Scott
Robert Nowak

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The Q-function is a convenient way to express right-tail probabilities for normal (Gaussian) random variables. For $x \in \mathbb{R}$, $Q(x)$ is defined as the probability that a standard normal random variable (zero mean, unit variance) exceeds x:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$$

Q is a mapping from $\mathbb{R}$ to $\{0, 1\}$. One may also define $Q(-\infty) = 1$ and $Q(\infty) = 0$.

ASIDE: Some authors define the Q-function in different ways. One alternative is to define $Q(x) = F(x) - F(0)$. This definition is discussed at MathWorld[1] .

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Figure 1: $Q(x)$ is represented by the shaded region.

Since $Q(x)$ is monotonically decreasing, it has a well-defined inverse $Q^{-1} : \{0, 1\} \to \mathbb{R}$.

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Figure 2: A plot of $Q(x)$

If $F(x)$ denotes the cumulative distribution function of a standard normal, then clearly $Q(x) = 1 - F(x)$. For this reason Q is also called the **complementary cumulative distribution function**. The Q-function is useful because the tail probability cannot be evaluated symbolically, and so $Q(x)$ offers a concise notation for this integral. It is similar to the gamma and beta functions in this respect.

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[1]http://mathworld.wolfram.com/NormalDistributionFunction.html

1 Arbitrary Mean and Variance

The Q-function is also useful for expressing right-tail probabilities of nonstandard normal variates. If

$$X \sim \mathcal{N}\left(\mu, \sigma^2\right)$$

then

$$\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$$

To express $Pr[X > \gamma]$ in terms of Q, where $\gamma \in \mathbb{R}$, define $\eta = \frac{\gamma-\mu}{\sigma}$. Then

$$\begin{aligned} Pr[X > \gamma] &= Pr[X > \eta\sigma + \mu] \\ &= Pr\left[\frac{X-\mu}{\sigma} > \eta\right] \\ &= Q(\eta) \\ &= Q\left(\frac{\gamma-\mu}{\sigma}\right) \end{aligned} \tag{1}$$

2 Relation to Erf and Erfinv

The erf function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Both erf and its inverse, erfinv, are built into many common mathematical software packages such as Mathematica and Matlab. Therefore, they can be used to numerically evaluate Q and Q^{-1}. By a change of variables, we have

$$Q(x) = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$$

and

$$Q^{-1}(y) = \sqrt{2}\operatorname{erfinv}(1-2y)$$

3 Approximations

One approximation that is sometimes useful for x away from zero is

$$Q(x) \simeq \frac{1}{x\sqrt{2\pi}} e^{\frac{-1}{2}x^2}$$