RANDOM PARAMETERS*

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When we know the density of θ , the likelihood function can be expressed as

$$p_{r \mid \mathcal{M}_{i}}(r) = \int p_{r \mid \mathcal{M}_{i}\theta}(r) p_{\theta}(\theta) d\theta$$

and the likelihood ratio in the random parameter case becomes

$$\Lambda(r) = \frac{\int p_{r \mid \mathcal{M}_{i}\theta}(r) p_{\theta}(\theta) d\theta}{\int p_{r \mid \mathcal{M}_{i}\theta}(r) p_{\theta}(\theta) d\theta}$$

Unfortunately, there are many examples where either the integrals involved are intractable or the sufficient statistic is virtually the same as the likelihood ratio, which can be difficult to compute.

Example 1

A simple, but interesting, example that results in a computable answer occurs when the mean of Gaussian random variables is either zero (model 0) or is $\pm (m)$ with equal probability (hypothesis 1). The second hypothesis means that a non-zero mean is present, but its sign is not known. We are therefore stating that if hypothesis one is in fact valid, the mean has fixed sign for each observation - what is random is its a priori value. As before, L statistically independent observations are made.

$$\mathcal{M}_0: r \sim \mathcal{N}\left(0, \sigma^2 I\right)$$

$$\mathcal{M}_{1}: r \sim \mathcal{N}\left(m, \sigma^{2}I\right)$$
$$m = \begin{cases} \begin{pmatrix} m \\ \cdots \\ m \end{pmatrix} & \operatorname{Prob} 1/2 \\ \begin{pmatrix} -m \\ \cdots \\ -m \end{pmatrix} & \operatorname{Prob} 1/2 \end{cases}$$

The numerator of the likelihood ratio is the sum of two Gaussian densities weighted by 1/2 (the *a priori* probability values), one having a positive mean, the other negative. The likelihood ratio, after simple cancellation of common terms, becomes

$$\Lambda\left(r\right) = \frac{1}{2}e^{\frac{2m\sum_{l=0}^{L-1}r_l - Lm^2}{2\sigma^2}} + \frac{1}{2}e^{\frac{-2m\sum_{l=0}^{L-1}r_l - Lm^2}{2\sigma^2}}$$

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and the decision rule takes the form

$$\cosh\left(\frac{m}{\sigma^2}\sum_{l=0}^{L-1}r_l\right) \underset{\mathcal{M}_0}{\overset{\mathcal{M}_1}{\gtrless}} \left(\eta e^{\frac{Lm^2}{2\sigma^2}}\right)$$

where $\cosh(x)$ is the **hyperbolic cosine** given simply as $\frac{e^x + e^{-x}}{2}$. As this quantity is an even function, the sign of its argument has no effect on the result. The decision rule can be written more simply as

$$\left|\sum_{l=0}^{L-1} r_l\right| \underset{\mathcal{M}_0}{\overset{\mathcal{M}_1}{\gtrless}} \left(\frac{\sigma^2}{|m|} \operatorname{arccosh}\left(\eta e^{\frac{Lm^2}{2\sigma^2}}\right)\right)$$

The sufficient statistic equals the **magnitude** of the sum of the observations in this case. While the right side of this expression, which equals γ , is complicated, it need only be computed once. Calculation of the performance probabilities can be complicated; in this case, the false-alarm probability is easy to find and the others more difficult.