Random Parameters*

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When we know the density of θ, the likelihood function can be expressed as

$$p_{r\,|\,\mathcal{M}_i}(r) = \int p_{r\,|\,\mathcal{M}_i\theta}(r)\, p_{\theta}(\theta)\, d\theta$$

and the likelihood ratio in the random parameter case becomes

$$\Lambda(r) = \frac{\int p_{r\,|\,\mathcal{M}_i\theta}(r)\, p_{\theta}(\theta)\, d\theta}{\int p_{r\,|\,\mathcal{M}_i\theta}(r)\, p_{\theta}(\theta)\, d\theta}$$

Unfortunately, there are many examples where either the integrals involved are intractable or the sufficient statistic is virtually the same as the likelihood ratio, which can be difficult to compute.

Example 1

A simple, but interesting, example that results in a computable answer occurs when the mean of Gaussian random variables is either zero (model 0) or is $\pm(m)$ with equal probability (hypothesis 1). The second hypothesis means that a non-zero mean is present, but its sign is not known. We are therefore stating that if hypothesis one is in fact valid, the mean has fixed sign for each observation - what is random is its *a priori* value. As before, L statistically independent observations are made.

$$\mathcal{M}_0 : r \sim \mathcal{N}\left(0, \sigma^2 I\right)$$

$$\mathcal{M}_1 : r \sim \mathcal{N}\left(m, \sigma^2 I\right)$$

$$m = \begin{cases} \begin{pmatrix} m \\ \dots \\ m \end{pmatrix} & \text{Prob } 1/2 \\ \begin{pmatrix} -m \\ \dots \\ -m \end{pmatrix} & \text{Prob } 1/2 \end{cases}$$

The numerator of the likelihood ratio is the sum of two Gaussian densities weighted by 1/2 (the *a priori* probability values), one having a positive mean, the other negative. The likelihood ratio, after simple cancellation of common terms, becomes

$$\Lambda(r) = \frac{1}{2} e^{\frac{2m\sum_{l=0}^{L-1} r_l - Lm^2}{2\sigma^2}} + \frac{1}{2} e^{\frac{-2m\sum_{l=0}^{L-1} r_l - Lm^2}{2\sigma^2}}$$

*Version 1.1: Aug 27, 2003 10:27 am -0500 and the decision rule takes the form

$$\cosh\left(\frac{m}{\sigma^2}\sum_{l=0}^{L-1} r_l\right) \underset{\mathcal{M}_0}{\overset{\mathcal{M}_1}{\gtrless}} \left(\eta e^{\frac{Lm^2}{2\sigma^2}}\right)$$

where $\cosh(x)$ is the **hyperbolic cosine** given simply as $\frac{e^x+e^{-x}}{2}$. As this quantity is an even function, the sign of its argument has no effect on the result. The decision rule can be written more simply as

$$|\sum_{l=0}^{L-1} r_l| \underset{\mathcal{M}_0}{\overset{\mathcal{M}_1}{\gtrless}} \left(\frac{\sigma^2}{|m|}\text{arccosh}\left(\eta e^{\frac{Lm^2}{2\sigma^2}}\right)\right)$$

The sufficient statistic equals the **magnitude** of the sum of the observations in this case. While the right side of this expression, which equals γ, is complicated, it need only be computed once. Calculation of the performance probabilities can be complicated; in this case, the false-alarm probability is easy to find and the others more difficult.