Differential Entropy*

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Abstract

In this module we consider differential entropy.

Consider the entropy of **continuous** random variables. Whereas the (normal) entropy is the entropy of a **discrete** random variable, the differential entropy is the entropy of a continuous random variable.

1 Differential Entropy

Definition 1: Differential entropy

The differential entropy $h(X)$ of a continuous random variable X with a pdf $f(x)$ is defined as

$$h(X) = -\int_{-\infty}^{\infty} f(x) \log f(x)\, dx \tag{1}$$

Usually the logarithm is taken to be base 2, so that the unit of the differential entropy is bits/symbol. Note that is the discrete case, $h(X)$ depends only on the pdf of X. Finally, we note that the differential entropy is the expected value of $-\log f(x)$, i.e.,

$$h(X) = -E(\log f(x)) \tag{2}$$

Now, consider a calculating the differential entropy of some random variables.

Example 1

Consider a uniformly distributed random variable X from c to $c+\Delta$. Then its density is $\frac{1}{\Delta}$ from c to $c+\Delta$, and zero otherwise.

We can then find its differential entropy as follows,

$$\begin{aligned} h(X) &= -\int_c^{c+\Delta} \frac{1}{\Delta} \log \frac{1}{\Delta}\, dx \\ &= \log \Delta \end{aligned} \tag{3}$$

Note that by making Δ arbitrarily small, the differential entropy can be made arbitrarily negative, while taking Δ arbitrarily large, the differential entropy becomes arbitrarily positive.

Example 2

Consider a normal distributed random variable X, with mean m and variance σ^2. Then its density is $\sqrt{\frac{1}{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$.

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We can then find its differential entropy as follows, first calculate $-\log f(x)$:

$$-\log f(x) = \frac{1}{2}\log\left(2\pi\sigma^2\right) + \log e\frac{(x-m)^2}{2\sigma^2} \tag{4}$$

Then since $E\left((X-m)^2\right) = \sigma^2$, we have

$$\begin{aligned} h(X) &= \tfrac{1}{2}\log\left(2\pi\sigma^2\right) + \tfrac{1}{2}\log e \\ &= \tfrac{1}{2}\log\left(2\pi e\sigma^2\right) \end{aligned} \tag{5}$$

2 Properties of the differential entropy

In the section we list some properties of the differential entropy.

- The differential entropy can be negative
- $h(X+c) = h(X)$, that is translation **does not** change the differential entropy.
- $h(aX) = h(X) + \log|a|$, that is scaling **does** change the differential entropy.

The first property is seen from both Example 1 and Example 2. The two latter can be shown by using (1).