Connexions module: m11840

DIFFERENTIAL ENTROPY*

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Abstract

In this module we consider differential entropy.

Consider the entropy of **continuous** random variables. Whereas the (normal) entropy¹ is the entropy of a **discrete** random variable, the differential entropy is the entropy of a continuous random variable.

1 Differential Entropy

Definition 1: Differential entropy

The differential entropy h(X) of a continuous random variable X with a pdf f(x) is defined as

$$h(X) = -\int_{-\infty}^{\infty} f(x) \log f(x) dx \tag{1}$$

Usually the logarithm is taken to be base 2, so that the unit of the differential entropy is bits/symbol. Note that is the discrete case, h(X) depends only on the pdf of X. Finally, we note that the differential entropy is the expected value of $-\log f(x)$, i.e.,

$$h(X) = -E(\log f(x)) \tag{2}$$

Now, consider a calculating the differential entropy of some random variables.

Example 1

Consider a uniformly distributed random variable X from c to $c + \Delta$. Then its density is $\frac{1}{\Delta}$ from c to $c + \Delta$, and zero otherwise.

We can then find its differential entropy as follows,

$$h(X) = -\int_{c}^{c+\Delta} \frac{1}{\Delta} \log \frac{1}{\Delta} dx$$

$$= \log \Delta$$
(3)

Note that by making Δ arbitrarily small, the differential entropy can be made arbitrarily negative, while taking Δ arbitrarily large, the differential entropy becomes arbitrarily positive.

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^{1&}quot;Entropy" http://cnx.org/content/m11839/latest/

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Example 2

Consider a normal distributed random variable X, with mean m and variance σ^2 . Then its density is $\sqrt{\frac{1}{2\pi\sigma^2}}e^{-\frac{(x-m)^2}{2\sigma^2}}$.

We can then find its differential entropy as follows, first calculate $-\log f(x)$:

$$-\log f(x) = \frac{1}{2}\log(2\pi\sigma^2) + \log(x-m)^2$$
(4)

Then since $E\left(\left(X-m\right)^2\right)=\sigma^2$, we have

$$h(X) = \frac{1}{2}\log(2\pi\sigma^2) + \frac{1}{2}\log e$$

= $\frac{1}{2}\log(2\pi e\sigma^2)$ (5)

2 Properties of the differential entropy

In the section we list some properties of the differential entropy.

- The differential entropy can be negative
- h(X+c) = h(X), that is translation **does not** change the differential entropy.
- $h(aX) = h(X) + \log|a|$, that is scaling **does** change the differential entropy.

The first property is seen from both Example 1 and Example 2. The two latter can be shown by using (1).