Signal Size and Norms*

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Abstract

A module concerning the size of a signal, more specifically norms.

1 Introduction

The "size" of a signal would involve some notion of its strength. We use the mathematical concept of the norm to quantify this concept for both continuous-time and discrete-time signals. As there are several types of norms that can be defined for signals, there are several different conceptions of signal size.

2 Signal Energy

2.1 Infinite Length, Continuous Time Signals

The most commonly encountered notion of the energy of a signal defined on $\mathbb{R}$ is the L_2 norm defined by the square root of the integral of the square of the signal, for which the notation

$$||f||_2 = \left(\int_{-\infty}^{\infty} |f(t)|^2 dt\right)^{1/2}. \tag{1}$$

However, this idea can be generalized through definition of the L_p norm, which is given by

$$||f||_p = \left(\int_{-\infty}^{\infty} |f(t)|^p dt\right)^{1/p} \tag{2}$$

for all $1 \leq p < \infty$. Because of the behavior of this expression as p approaches ∞, we furthermore define

$$||f||_\infty = \sup_{t\in\mathbb{R}} |f(t)|, \tag{3}$$

which is the least upper bound of $|f(t)|$. A signal f is said to belong to the vector space $L_p(\mathbb{R})$ if $||f||_p < \infty$.

Example 1

For example, consider the function defined by

$$f(t) = \begin{cases} 1/t & 1 \leq t \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

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The L_1 norm is

$$||f||_1 = \int_{-\infty}^{\infty} |f(t)|dt = \int_{-\infty}^{\infty} \frac{1}{t} dt = \infty. \tag{5}$$

The L_2 norm is

$$||f||_2 = \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^{1/2} = \left(\int_{-\infty}^{\infty} \frac{1}{t^2} dt \right)^{1/2} = 1. \tag{6}$$

The L_∞ norm is

$$||f||_\infty = \sup_{t \in \mathbb{R}} |f(t)| = \sup_{t \in \mathbb{R}[1,\infty)} \frac{1}{t} = 1. \tag{7}$$

2.2 Finite Length, Continuous Time Signals

The most commonly encountered notion of the energy of a signal defined on $\mathbb{R}[a,b]$ is the L_2 norm defined by the square root of the integral of the square of the signal, for which the notation

$$||f||_2 = \left(\int_a^b |f(t)|^2 dt \right)^{1/2}. \tag{8}$$

However, this idea can be generalized through definition of the L_p norm, which is given by

$$||f||_p = \left(\int_a^b |f(t)|^p dt \right)^{1/p} \tag{9}$$

for all $1 \le p < \infty$. Because of the behavior of this expression as p approaches ∞, we furthermore define

$$||f||_\infty = \sup_{t \in \mathbb{R}[a,b]} |f(t)|, \tag{10}$$

which is the least upper bound of $|f(t)|$. A signal f is said to belong to the vector space $L_p(\mathbb{R}[a,b])$ if $||f||_p < \infty$. The periodic extension of such a signal would have infinite energy but finite power.

Example 2

For example, consider the function defined on $\mathbb{R}[-5,3]$ by

$$f(t) = \begin{cases} t & -5 < t < 3 \\ 0 & \text{otherwise} \end{cases}. \tag{11}$$

The L_1 norm is

$$||f||_1 = \int_{-5}^{3} |f(t)|dt = \int_{-5}^{3} |t| dt = 17. \tag{12}$$

The L_2 norm is

$$||f||_2 = \left(\int_{-5}^{3} |f(t)|^2 dt \right)^{1/2} = \left(\int_{-5}^{3} |t|^2 dt \right)^{1/2} \approx 7.12 \tag{13}$$

The L_∞ norm is

$$||f||_\infty = \underset{t \in \mathbb{R}[-5,3]}{sup} |t| = 5. \quad (14)$$

2.3 Infinite Length, Discrete Time Signals

The most commonly encountered notion of the energy of a signal defined on $\mathbb{Z}$ is the l_2 norm defined by the square root of the sumation of the square of the signal, for which the notation

$$||f||_2 = \left(\sum_{n=-\infty}^{\infty} |f(n)|^2 \right)^{1/2}. \quad (15)$$

However, this idea can be generalized through definition of the l_p norm, which is given by

$$||f||_p = \left(\sum_{n=-\infty}^{\infty} |f(n)|^p \right)^{1/p} \quad (16)$$

for all $1 \leq p < \infty$. Because of the behavior of this expression as p approaches ∞, we furthermore define

$$||f||_\infty = \underset{n \in \mathbb{Z}}{sup} |f(n)|, \quad (17)$$

which is the least upper bound of $|f(n)|$. A signal f is said to belong to the vector space $l_p(\mathbb{Z})$ if $||f||_p < \infty$.

Example 3

For example, consider the function defined by

$$f(n) = \{ \begin{array}{cc} 1/n & 1 \leq n \\ 0 & \text{otherwise} \end{array}. \quad (18)$$

The l_1 norm is

$$||f||_1 = \sum n = -\infty^\infty |f(n)| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \quad (19)$$

The l_2 norm is

$$||f||_2 = \left(\sum_{n=-\infty}^{\infty} |f(n)|^2 \right)^{1/2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} = \frac{\pi\sqrt{6}}{6} \quad (20)$$

The l_∞ norm is

$$||f||_\infty = \underset{n \in \mathbb{Z}}{sup} |f(n)| = \underset{n \in \mathbb{Z}[1,\infty)}{sup} \frac{1}{n} = 1. \quad (21)$$

2.4 Finite Length, Discrete Time Signals

The most commonly encountered notion of the energy of a signal defined on $\mathbb{Z}[a,b]$ is the l_2 norm defined by the square root of the sumation of the square of the signal, for which the notation

$$||f||_2 = \left(\sum_{n=a}^{b} |f(n)|^2\right)^{1/2}. \tag{22}$$

However, this idea can be generalized through definition of the l_p norm, which is given by

$$||f||_p = \left(\sum_{n=a}^{b} |f(n)|^p\right)^{1/p} \tag{23}$$

for all $1 \leq p < \infty$. Because of the behavior of this expression as p approaches ∞, we furthermore define

$$||f||_\infty = \sup_{n\in\mathbb{Z}[a,b]} |f(n)|, \tag{24}$$

which is the least upper bound of $|f(n)|$. In this case, this least upper bound is simply the maximum value of $|f(n)|$. A signal f is said to belong to the vector space $l_p(\mathbb{Z}[a,b])$ if $||f||_p < \infty$. The periodic extension of such a signal would have infinite energy but finite power.

Example 4

For example, consider the function defined on $\mathbb{Z}[-5,3]$ by

$$f(n) = \{ \begin{array}{cc} n & -5 < n < 3 \\ 0 & \text{otherwise} \end{array}. \tag{25}$$

The l_1 norm is

$$||f||_1 = \sum_{n=-5}^{3} |f(n)| = \sum -5^3 |n| = 21. \tag{26}$$

The l_2 norm is

$$||f||_2 = \left(\sum_{-5}^{3} |f(n)|^2\right)^{1/2} = \left(\sum_{-5}^{3} |n|^2 dt\right)^{1/2} \approx 8.31 \tag{27}$$

The l_∞ norm is

$$||f||_\infty = \sup_{n\in\mathbb{Z}[-5,3]} |f(n)| = 5. \tag{28}$$

3 Signal Norms Summary

The notion of signal size or energy is formally addressed through the mathematical concept of norms. There are many types of norms that can be defined for signals, some of the most important of which have been discussed here. For each type norm and each type of signal domain (continuous or discrete, and finite or infinite) there are vector spaces defined for signals of finite norm. Finally, while nonzero periodic signals have infinite energy, they have finite power if their single period units have finite energy.