Simple and Compound Pendula*

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1 The Simple Pendulum

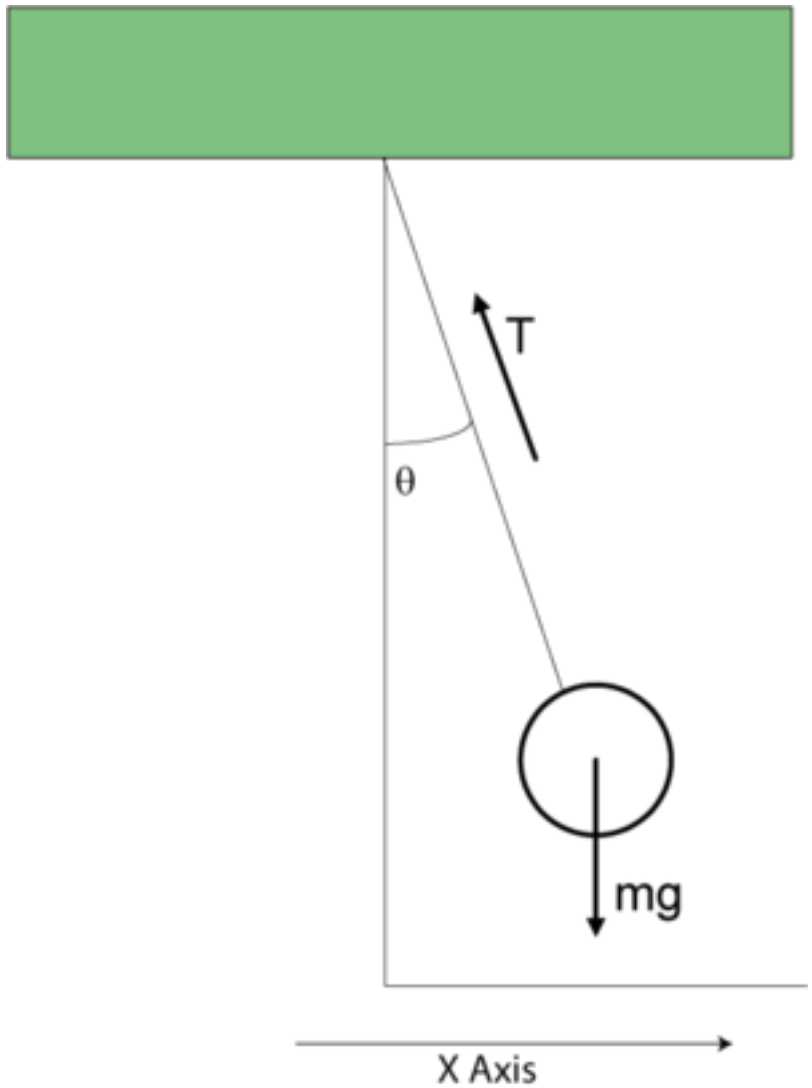

Figure 1: A simple pendulum.

Shown is a simple pendulum which has a mass m that is displaced by an angle θ. There is tension ($\vec{T}$) in the string which acts from the mass to the anchor point. The weight of the mass is $m\ \vec{g}$ and the tension in the string is $T = mg\cos\theta$. There is a tangential restoring force $= -mg\sin\theta$. If we approximate that θ is small (we have to make this approximation or else we can not solve the problem analytically) then $\sin\theta \approx \theta$ and $x = l\theta$. (note that $\sin\theta$ is only approximately equal to $\frac{x}{l}$ because x is the distance along the x axis) so

that we can write:

$$\begin{aligned} F = ma = m\ddot{x} \\ = -mg\sin\theta \\ \approx -mg\theta \\ \approx -mg\frac{x}{l} \end{aligned}$$

or

$$\ddot{x} + \frac{g}{l}x = 0$$

(Note that We should immediately recognize that this is the equation for simple harmonic motion (SHM) with

$$\omega = \sqrt{\frac{g}{l}}.$$

We could take another approach and use angular momentum to solve the problem. Recall that:

$$L = I\omega = I\dot{\theta}$$

$$I = ml^2.$$

Also recall that the torque is the time derivative of the angular momentum so that:

$$\begin{aligned} \vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} \\ -lmg\theta = I\ddot{\theta} \end{aligned}$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

Again we would recognize that this is simple harmonic motion with

$$\omega = \sqrt{\frac{g}{l}}.$$

2 The Compound Pendulum

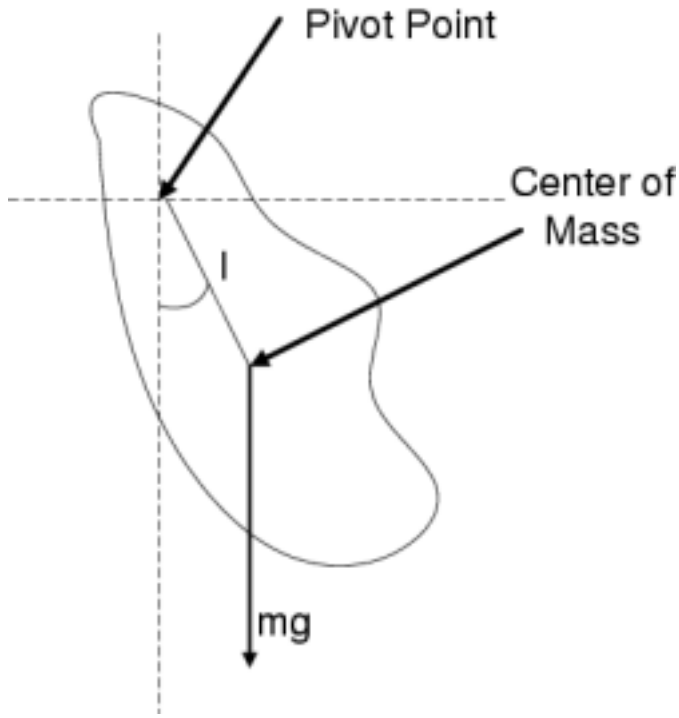

Figure 2: A compound pendulum.

The compound pendulum is another interesting example of a pendulum that undergoes simple harmonic motion. For an extended body then one uses the center of mass and the moment of inertia. Use the center of mass, the moment of inertia and the Torque (angular force) $\vec{\tau}=\vec{r}\times\vec{F}$

$$\tau = r \times F$$
$$I\ddot{\theta} = -lmg\sin\theta \approx -lmg\theta$$
$$\ddot{\theta} + \frac{lmg}{I}\theta = 0$$

So again we get SHM now with

$$\omega^2 = \frac{lmg}{I}$$

One sees that this formalism can be applied to the simple pendulum (ignore the string and one can consider the ball a point mass). The moment of inertia is ml^2. So we get

$$\omega^2 = \frac{lmg}{ml^2} = \frac{g}{l}$$

which is just what we got before for the simple pendulum. We could write the equation of motion for a simple pendulum as:

$$\theta = Ae^{i(\omega t+\phi_0)}$$

where ϕ_0 is determined by initial conditions.

A discussion of the Pendulum and Simple Harmonic Oscillator can be found at http://monet.physik.unibas.ch/∼elmer/pendulum/index.html[1]

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