Frequency Sampling Design Method for FIR filters*

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Given a desired frequency response, the frequency sampling design method designs a filter with a frequency response **exactly** equal to the desired response at a particular set of frequencies ω_k.

Procedure

$$\forall k, k = [o, 1, \dots, N-1] : \left(H_d(\omega_k) = \sum_{n=0}^{M-1} h(n) e^{-(i\omega_k n)} \right) \quad (1)$$

NOTE: Desired Response must incluce linear phase shift (if linear phase is desired)

Exercise 1 ***(Solution on p. 4.)***

What is $H_d(\omega)$ for an ideal lowpass filter, cotoff at ω_c?

NOTE: This set of linear equations can be written in matrix form

$$H_d(\omega_k) = \sum_{n=0}^{M-1} h(n) e^{-(i\omega_k n)} \quad (2)$$

$$\begin{pmatrix} H_d(\omega_0) \\ H_d(\omega_1) \\ \vdots \\ H_d(\omega_{N-1}) \end{pmatrix} = \begin{pmatrix} e^{-(i\omega_0 0)} & e^{-(i\omega_0 1)} & \dots & e^{-(i\omega_0 (M-1))} \\ e^{-(i\omega_1 0)} & e^{-(i\omega_1 1)} & \dots & e^{-(i\omega_1 (M-1))} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-(i\omega_{M-1} 0)} & e^{-(i\omega_{M-1} 1)} & \dots & e^{-(i\omega_{M-1} (M-1))} \end{pmatrix} \begin{pmatrix} h(0) \\ h(1) \\ \vdots \\ h(M-1) \end{pmatrix} \quad (3)$$

or

$$H_d = Wh$$

So

$$h = W^{-1} H_d \quad (4)$$

NOTE: W is a square matrix for $N = M$, and invertible as long as $\omega_i \neq \omega_j + 2\pi l$, $i \neq j$

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1 Important Special Case

What if the frequencies are equally spaced between 0 and 2π, i.e. $\omega_k = \frac{2\pi k}{M} + \alpha$

Then

$$H_d(\omega_k) = \sum_{n=0}^{M-1} h(n) e^{-\left(i\frac{2\pi kn}{M}\right)} e^{-(i\alpha n)} = \sum_{n=0}^{M-1} \left(h(n) e^{-(i\alpha n)}\right) e^{-\left(i\frac{2\pi kn}{M}\right)} = \text{DFT!}$$

so

$$h(n) e^{-(i\alpha n)} = \frac{1}{M} \sum_{k=0}^{M-1} H_d(\omega_k) e^{i\frac{2\pi nk}{M}}$$

or

$$h[n] = \frac{e^{i\alpha n}}{M} \sum_{k=0}^{M-1} H_d[\omega_k] e^{i\frac{2\pi nk}{M}} = e^{i\alpha n}\text{IDFT}[H_d[\omega_k]]$$

2 Important Special Case #2

$h[n]$ symmetric, linear phase, and has real coefficients. Since $h[n] = h[M-n]$, there are only $\frac{M}{2}$ degrees of freedom, and only $\frac{M}{2}$ linear equations are required.

$$\begin{aligned} H[\omega_k] &= \sum_{n=0}^{M-1} h[n] e^{-(i\omega_k n)} \\ &= \begin{cases} \sum_{n=0}^{\frac{M}{2}-1} h[n]\left(e^{-(i\omega_k n)} + e^{-(i\omega_k(M-n-1))}\right) & \text{if M even} \\ \sum_{n=0}^{M-\frac{3}{2}} h[n]\left(e^{-(i\omega_k n)} + e^{-(i\omega_k(M-n-1))}\right)\left(h\left[\frac{M-1}{2}\right] e^{-\left(i\omega_k\frac{M-1}{2}\right)}\right) & \text{if M odd} \end{cases} \\ &= \begin{cases} e^{-\left(i\omega_k\frac{M-1}{2}\right)} 2\sum_{n=0}^{\frac{M}{2}-1} h[n]\cos\left(\omega_k\left(\frac{M-1}{2}-n\right)\right) & \text{if M even} \\ e^{-\left(i\omega_k\frac{M-1}{2}\right)} 2\sum_{n=0}^{M-\frac{3}{2}} h[n]\cos\left(\omega_k\left(\frac{M-1}{2}-n\right)\right) + h\left[\frac{M-1}{2}\right] & \text{if M odd} \end{cases} \end{aligned} \quad (5)$$

Removing linear phase from both sides yields

$$A(\omega_k) = \begin{cases} 2\sum_{n=0}^{\frac{M}{2}-1} h[n]\cos\left(\omega_k\left(\frac{M-1}{2}-n\right)\right) & \text{if M even} \\ 2\sum_{n=0}^{M-\frac{3}{2}} h[n]\cos\left(\omega_k\left(\frac{M-1}{2}-n\right)\right) + h\left[\frac{M-1}{2}\right] & \text{if M odd} \end{cases}$$

Due to symmetry of response for real coefficients, only $\frac{M}{2}\omega_k$ on $\omega \in [0,\pi)$ need be specified, with the frequencies $-\omega_k$ thereby being implicitly defined also. Thus we have $\frac{M}{2}$ **real-valued** simultaneous linear equations to solve for $h[n]$.

2.1 Special Case 2a

$h[n]$ symmetric, odd length, linear phase, real coefficients, and ω_k equally spaced: $\forall k, 0 \leq k \leq M-1 : \left(\omega_k = \frac{n\pi k}{M}\right)$

$$\begin{aligned} h[n] &= \text{IDFT}[H_d(\omega_k)] \\ &= \frac{1}{M}\sum_{k=0}^{M-1} A(\omega_k) e^{-\left(i\frac{2\pi k}{M}\right)\frac{M-1}{2}} e^{i\frac{2\pi nk}{M}} \\ &= \frac{1}{M}\sum_{k=0}^{M-1} A(k) e^{i\left(\frac{2\pi k}{M}\left(n-\frac{M-1}{2}\right)\right)} \end{aligned} \quad (6)$$

To yield real coefficients, $A(\omega)$ mus be symmetric

$$(A(\omega) = A(-\omega)) \Rightarrow (A[k] = A[M-k])$$

$$\begin{aligned} h[n] &= \frac{1}{M}\left(A(0) + \sum_{k=1}^{\frac{M-1}{2}} A[k]\left(e^{i\frac{2\pi k}{M}\left(n-\frac{M-1}{2}\right)} + e^{-\left(i2\pi k\left(n-\frac{M-1}{2}\right)\right)}\right)\right) \\ &= \frac{1}{M}\left(A(0) + 2\sum_{k=1}^{\frac{M-1}{2}} A[k]\cos\left(\frac{2\pi k}{M}\left(n-\frac{M-1}{2}\right)\right)\right) \\ &= \frac{1}{M}\left(A(0) + 2\sum_{k=1}^{\frac{M-1}{2}} A[k](-1)^k\cos\left(\frac{2\pi k}{M}\left(n+\frac{1}{2}\right)\right)\right) \end{aligned} \quad (7)$$

Simlar equations exist for even lengths, anti-symmetric, and $\alpha = \frac{1}{2}$ filter forms.

3 Comments on frequency-sampled design

This method is simple conceptually and very efficient for equally spaced samples, since $h[n]$ can be computed using the IDFT.

$H(\omega)$ for a frequency sampled design goes **exactly** through the sample points, but it may be very far off from the desired response for $\omega \neq \omega_k$. This is the main problem with frequency sampled design.

Possible solution to this problem: specify more frequency samples than degrees of freedom, and minimize the total error in the frequency response at all of these samples.

4 Extended frequency sample design

For the samples $H(\omega_k)$ where $0 \leq k \leq M-1$ and $N > M$, find $h[n]$, where $0 \leq n \leq M-1$ minimizing $\| H_d(\omega_k) - H(\omega_k) \|$

For $\| l \|_\infty$ norm, this becomes a linear programming problem (standard packages availble!)

Here we will consider the $\| l \|_2$ norm.

To minimize the $\| l \|_2$ norm; that is, $\sum_{n=0}^{N-1} |H_d(\omega_k) - H(\omega_k)|$, we have an overdetermined set of linear equations:

$$\begin{pmatrix} e^{-(i\omega_0 0)} & \dots & e^{-(i\omega_0(M-1))} \\ \vdots & \vdots & \vdots \\ e^{-(i\omega_{N-1}0)} & \dots & e^{-(i\omega_{N-1}(M-1))} \end{pmatrix} h = \begin{pmatrix} H_d(\omega_0) \\ H_d(\omega_1) \\ \vdots \\ H_d(\omega_{N-1}) \end{pmatrix}$$

or

$$Wh = H_d$$

The minimum error norm solution is well known to be $h = \left(\overline{W}W\right)^{-1}\overline{W}H_d$; $\left(\overline{W}W\right)^{-1}\overline{W}$ is well known as the pseudo-inverse matrix.

> NOTE: Extended frequency sampled design discourages radical behavior of the frequency response between samples for sufficiently closely spaced samples. However, the actual frequency response may no longer pass exactly through **any** of the $H_d(\omega_k)$.

Solutions to Exercises in this Module

Solution to Exercise (p. 1)

$$\begin{cases} e^{-\left(i\omega\frac{M-1}{2}\right)} & \text{if } -\omega_c \leq \omega \leq \omega_c \\ 0 \text{ if } (-\pi \leq \omega < -\omega_c) \vee (\omega_c < \omega \leq \pi) \end{cases}$$